

**BOUNDEDNESS AND CONTINUITY OF THE
FUNDAMENTAL OPERATIONS ON DISTRIBUTIONS
HAVING A SPECIFIED WAVE FRONT SET.**

(WITH A COUNTER EXAMPLE BY SEMYON ALESKER)

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ABSTRACT. The pull-back, push-forward and multiplication of smooth functions can be extended to distributions if their wave front set satisfies some conditions. Thus, it is natural to investigate the topological properties of these operations between spaces \mathcal{D}'_Γ of distributions having a wave front set included in a given closed cone Γ of the cotangent space. As discovered by S. Alesker, the pull-back is not continuous for the usual topology on \mathcal{D}'_Γ , and the tensor product is not separately continuous. In this paper, the pseudo-topology of \mathcal{D}'_Γ , defined by Hörmander is shown to be a bornology and a new topology is defined for which the pull-back and push-forward are continuous, the tensor product, the multiplication and the convolution product of distributions are hypocontinuous.

1. Introduction

The motivation of our work comes from the renormalization of QFT in curved space times, indeed the question addressed in this paper cannot be avoided in this context and also the technical results of this paper form the core of the proof that perturbative quantum field theories are renormalizable on curved space times [11, 10].

Since L. Schwartz, we know that the tensor product of distributions is continuous [34, p. 110] and the product of a distribution by a smooth function is hypocontinuous [34, p. 119] (see definition 4.2), although it is not jointly continuous [28, 29].

Keywords: microlocal analysis, functional analysis, mathematical physics, renormalization.

However, in many applications (for instance the multiplication of distributions involved in quantum field theory), we cannot work with all distributions and we must consider the subsets \mathcal{D}'_Γ of distributions whose wave front set is included in some closed subset Γ of $\dot{T}^*\mathbb{R}^n = \{(x; \xi) \in T^*\mathbb{R}^n; \xi \neq 0\}$. In the following, Γ will always denote a **cone** i.e. $(x; \xi) \in \Gamma \implies (x; \lambda\xi) \in \Gamma$ for every $\lambda \in \mathbb{R}_{>0}$. Indeed the spaces \mathcal{D}'_Γ are widely used in microlocal analysis because wave front set conditions rule the fundamental operations on distributions (multiplication, pull-back, push-forward and restriction) which are known to be sequentially continuous [6].

Hörmander himself, who introduced the concept of a wave front set [23], equipped \mathcal{D}'_Γ with a *pseudo-topology* [23, p. 125], which is just a rule describing the convergence of sequences and not a topology. In particular, when Hörmander writes that the pullback is continuous [24, p. 263], he means “sequentially continuous”, precisely because he does not define any topology on \mathcal{D}'_Γ (in [23] Hörmander states explicitly that the fundamental operations are sequentially continuous).

Duistermaat’s famous lecture notes [13] are more ambiguous because they define a locally convex topology on \mathcal{D}'_Γ and they state that the pull-back [13, p. 19], the push-forward [13, p. 20] and the product of distributions [13, p. 21] are continuous maps. A similar confusion exists in the modern literature, where Heifetz states that the pull-back by a smooth map is continuous [20].

The purpose of the present paper is to clarify the situation, to determine the precise regularity of the operations with distributions and to define a new topology for which the fundamental operations have optimal continuity properties. Our paper is divided in two parts.

In the first part, we review Hörmander’s pseudo-topology and we show that the right framework to understand it is the natural bornology of \mathcal{D}'_Γ [9]. Indeed this concept, introduced by Bourbaki [2], is enough to control the sequential continuity in \mathcal{D}'_Γ . However, sequential continuity and bornology are not well adapted to applications in renormalization of QFT on curved space times and valuation theory [1]. For such applications, it is desirable to have available a topology on \mathcal{D}'_Γ for which the usual operations of pull-back and push-forward are continuous and the tensor product is hypocontinuous. And most importantly, we need to study situations where the maps depend smoothly on extra parameters so that pull-back and push-forward should be *uniformly continuous* w.r.t. the extra parameters. Duistermaat, in his lecture notes [13], proposed a locally convex topology on \mathcal{D}'_Γ . However, we shall present in subsection 3.1 a counterexample due

to S. Alesker showing that the tensor product is not separately continuous and the pull-back is not continuous for this topology.

This prompts us to introduce in the second part another, finer topology on \mathcal{D}'_Γ . This topology, called the *normal* topology of \mathcal{D}'_Γ by Dabrowski and Brouder [9], is defined in section 4. In this new topology, we prove in Theorem 5.5 that the tensor product is hypocontinuous. In sections 6 and 7, we use the functional properties of \mathcal{D}'_Γ [9], to give conceptually transparent proofs, without hard analytic estimates and using only the geometry of wave front sets, that:

- the pull-back by a smooth map is continuous (Proposition 6.1)
- the pull-back by a family of smooth maps depending smoothly on parameters is uniformly continuous (Theorem 6.9)
- the push-forward by a smooth map is also continuous (Theorem 7.3)
- the push-forward by a family of smooth maps (Theorem 7.4) depending smoothly on parameters is uniformly continuous
- the multiplication of distributions (Theorem 7.1) and the convolution product (Theorem 7.5) are hypocontinuous.

Finally in section 8, we discuss how the wave front set of distributions on manifolds can be defined in an intrinsic way.

In appendices, we prove important technical results concerning the covering of the complement of Γ , the topology of \mathcal{D}'_\emptyset and the fact that the additional seminorms used to define the topology of \mathcal{D}'_Γ can be taken to be countable.

FIRST PART

2. Sequential continuity in \mathcal{D}'_Γ

In his fundamental paper [23], Hörmander defines, for every open subset Ω of \mathbb{R}^n and every closed cone $\Gamma \subset \dot{T}^*\Omega$, the space $\mathcal{D}'_\Gamma = \{u \in \mathcal{D}'(\Omega); \text{WF}(u) \subset \Gamma\}$. He endows \mathcal{D}'_Γ with a *pseudo-topology*, a concept due to Choquet [7, 30] where conditions of convergence of sequences are given but no topology is defined (for example by a family of neighborhoods or of open sets). More precisely [24, p. 262],

DEFINITION 2.1. — *Let Ω be some open set in \mathbb{R}^n . We say that a sequence $(u_j) \in \mathcal{D}'_\Gamma(\Omega)$ converges to u in \mathcal{D}'_Γ in the sense of Hörmander iff*

- (i) (u_j) converges to u for the weak topology of $\mathcal{D}'(\Omega)$,
- (ii) for every $\chi \in \mathcal{D}(\Omega)$, every $N \in \mathbb{N}$ and every closed cone V in \mathbb{R}^n such that $\text{supp } \chi \times V \cap \Gamma = \emptyset$,

$$\sup_{k \in V} (1 + |k|)^N |\widehat{u_j \chi}(k) - \widehat{u \chi}(k)| \rightarrow 0 \text{ for } j \rightarrow \infty.$$

A map $f : \mathcal{D}_{\Gamma_1}(\Omega_1) \rightarrow \mathcal{D}_{\Gamma_2}(\Omega_2)$ is said to be *sequentially continuous* iff the image of a sequence (u_j) that converges to u is a sequence $(f(u_j))$ that converges to $f(u)$.

It was noticed by Hogbe-Nlend [21, p. 10] that a pseudo-topology can often be interpreted in terms of a bornology. In this section, we prove that this is the case for \mathcal{D}'_{Γ} and that a map is sequentially continuous iff it is bounded.

We give some basic definitions of bornological concepts.

2.1. Bornology

A bornology on X is a family of bounded subsets of X . The abstract definition is

DEFINITION 2.2. — A bornology on a set X is a family \mathcal{B} of subsets of X (called the bounded (sub)sets of X) such that: (i) every one-element subset of X belongs to \mathcal{B} ; (ii) if $A \in \mathcal{B}$ and $B \subset A$ then $B \in \mathcal{B}$ and (iii) if A and B are in \mathcal{B} then $A \cup B \in \mathcal{B}$.

These axioms are very natural properties of bounded sets: a point is bounded, a subset of a bounded set is bounded and the union of two bounded sets is bounded.

Among topological vector spaces, it is well known that the locally convex ones have nice properties. A similar concept holds for bornologies but we need first to define a disked hull:

DEFINITION 2.3. — A subset B of a vector space E over \mathbb{K} ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) is a disk iff $\lambda x + \mu y \in B$ whenever $x \in B$, $y \in B$, $\lambda \in \mathbb{K}$, $\mu \in \mathbb{K}$ and $|\lambda| + |\mu| \leq 1$. The disked hull of a subset B of a vector space E is the smallest disk of E containing B .

Now, a class of bornological vector space which has good properties is the class of convex bornological spaces.

DEFINITION 2.4. — Let E be a vector space over \mathbb{K} . A bornology \mathcal{B} on E is said to be a convex bornology if, for every A and B in \mathcal{B} and every

$\lambda \in \mathbb{K}$, the sets $A + B$, λA and the disked hull of A belong to \mathcal{B} . Then E or (E, \mathcal{B}) is called a convex bornological space.

Let E be a topological vector space, then by definition the bounded sets are the subsets B of E such that for any neighborhood V of $0 \in E$, there exists $t \geq 0$ such that for all λ with $|\lambda| \geq t$, $B \subset \lambda V$. In particular, if E is a locally convex space whose topology is defined by a family of seminorms $(p_{\alpha})_{\alpha \in I}$, the bounded sets of E are the sets that are bounded for every seminorm p_{α} and they form a natural bornology, called the von Neumann bornology of E .

It was proved in [9, Thm 33] that all families of seminorms on \mathcal{D}'_{Γ} which define topologies which are finer than the weak topology and coarser than the Mackey topology (see [3, p. IV.4] for the definition of the Mackey topology) give the same von Neumann bornology (i.e. define the same bounded sets). Hence this canonical von Neumann bornology will be called the bornology of \mathcal{D}'_{Γ} .

The interesting maps between two topological spaces are the continuous maps. The analogous objects for bornological spaces are the bounded maps:

DEFINITION 2.5. — *A map $f : E \rightarrow F$ between two bornological spaces is said to be bounded if the image of every bounded set of E is a bounded set of F .*

We will show that sequential continuity and boundedness are equivalent for \mathcal{D}'_{Γ} .

2.2. Bornological convergence

The concept of convergence relevant for bornological spaces was proposed in 1938 by Fichtenholz [15] and was elaborated by Mackey in his PhD thesis. This definition can be motivated as follows. In a normed space a sequence (x_n) converges to zero iff the sequence $(\|x_n\|)$ converges to zero [25, p. 9]. This happens iff there is a sequence (β_n) of positive real numbers tending to zero such that $\|x_n\| \leq \beta_n$ for every integer n . In other words, the sequence (x_n) converges to zero iff there is a sequence (β_n) of positive real numbers tending to zero such that $x_n \in \beta_n B(1)$, where $B(1) = \{y; \|y\| \leq 1\}$ is the unit ball. In a bornological space there is generally no unit ball and we must extend this definition by noticing that a sequence (x_n) converges to zero iff there is a sequence (α_n) of positive real numbers tending to zero and a bounded set B such that $x_n \in \alpha_n B$. Indeed, the boundedness of

B implies that there is a number M such that $\|b\| \leq M$ for all $b \in B$. Thus, $\|x_n\| \leq \beta_n$, where $\beta_n = M\alpha_n$ is a sequence of positive real numbers tending to zero.

We have now reached a definition that is valid in any convex bornological space [27, p. 12]:

DEFINITION 2.6. — *Let E be a convex bornological space. A sequence (x_n) in E is said to converge bornologically (or to Mackey-converge) to zero if there exists a bounded disk $B \subset E$ and a sequence (α_n) of positive real numbers tending to zero, such that $x_n \in \alpha_n B$ for every integer n .*

To ensure that the limit of a convergent sequence is unique we say that a convex bornological space is *separated* if the only vector space of \mathcal{B} is $\{0\}$. In other words, it is separated if no straight line is bounded. As a consequence [22, p. 28], a convex bornological space is separated iff every Mackey-convergent sequence has a unique limit. The von Neumann bornology of \mathcal{D}'_Γ is separated.

Hörmander convergence is the bornological (Mackey) convergence. The first link between pseudo-topology and bornology comes from the following

PROPOSITION 2.7. — *A sequence u_j in \mathcal{D}'_Γ converges to u in the sense of Hörmander iff it Mackey-converges to u for the bornology of \mathcal{D}'_Γ .*

The above result relates a topological notion of convergence with a bornological notion of convergence. We postpone the proof of Proposition 2.7 which is given in section 3. This brings us to the main result of this subsection, namely the identification between bounded maps and sequentially continuous maps:

PROPOSITION 2.8. — *A linear map $f : \mathcal{D}'_{\Gamma_1}(\Omega_1) \rightarrow \mathcal{D}'_{\Gamma_2}(\Omega_2)$ is sequentially continuous iff it is bounded.*

Proof. — We must show that the map f is bounded iff it transforms any convergent sequence of $\mathcal{D}'_{\Gamma_1}(\Omega_1)$ in the sense of Hörmander into a convergent sequence of $\mathcal{D}'_{\Gamma_2}(\Omega_2)$ in the sense of Hörmander. Because of Prop. 2.7, this is equivalent to saying that f transforms a Mackey-convergent sequence into a Mackey-convergent sequence. If E is a convex bornological space and F a locally convex space endowed with the corresponding von Neumann bornology, then a linear map from E to F is bounded iff the image of every Mackey-convergent sequence is a Mackey-convergent sequence. The direct sense, *bounded \rightarrow Mackey sequentially continuous*, is easy to prove since

a linear bounded map f sends bounded disked hulls to bounded disked hulls. We prove the converse by a contradiction argument. Assume f maps Mackey convergent sequences to Mackey convergent sequences but f is not bounded. Then

$\exists B \subset E$ bounded s.t. $\exists p_\alpha$ seminorm of F s.t. $\forall n, \exists x \in B, p_\alpha(f(x)) \geq n$.

Therefore we can find some sequence $(x_n)_n$ in B such that $p_\alpha(f(x_n)) \geq n$. Define the sequence $(\frac{x_n}{n})_{n \in \mathbb{N}}$ which is Mackey convergent in E since $\frac{x_n}{n} \in \frac{1}{n}D$ where D is the disked hull of B . By assumption, $f(\frac{x_n}{n}) \in F$ is Mackey convergent, therefore we should have $p_\alpha(f(\frac{x_n}{n})) \rightarrow 0$ but this is in contradiction with the estimate $p_\alpha(f(x_n)) \geq n \Leftrightarrow p_\alpha(f(\frac{x_n}{n})) \geq 1$.

Hence the proposition is proved. \square

We saw in section 2.1 that the bornology of \mathcal{D}'_Γ is compatible with many topologies (from the weak to the Mackey topology) which are all quasi-complete. Thus, every Cauchy sequence converges for these topologies and the space \mathcal{D}'_Γ is complete as a bornological space [22, p. 46].

2.3. Boundedness of the fundamental operations

It is very convenient to be able to manipulate bounded sets of distributions in \mathcal{D}'_Γ . For example, renormalization of quantum field theory in curved spacetimes requires to control the scaling of distributions, which is defined in terms of bounded sets of distributions [31].

By using Prop. 2.8, we can translate the results of sequential continuity of operations with distributions into statements about their boundedness. This is the object of the present section. In the sequel, for any open set $\Omega \subset \mathbb{R}^d$, we will denote by $\dot{T}^*\Omega$ the cotangent space $T^*\Omega$ minus the zero section.

2.3.1. Pull-back

Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets. For any smooth map $f : \Omega_1 \rightarrow \Omega_2$ we define the *pull-back operation* $f^* : C^\infty(\Omega_2) \rightarrow C^\infty(\Omega_1)$ by $u \mapsto f^*u := u \circ f$ and we define the set of normals of f by $N_f = \{(f(x); \eta) \in \Omega_2 \times \mathbb{R}^{d_2} ; \eta \circ df_x = 0\}$, where we abused notation by writing \mathbb{R}^{d_2} for its

dual $(\mathbb{R}^{d_2})^*$ and where

$$\begin{aligned} \eta \circ df_x &:= \sum_{j=1}^{d_2} \eta_j d(y^j \circ f)_x = \sum_{j=1}^{d_2} \eta_j dy^j \circ df_x \\ &= \sum_{j=1}^{d_2} \eta_j df_x^j = \sum_{j=1}^{d_2} \sum_{i=1}^{d_1} \eta_j \frac{\partial f^j}{\partial x^i} dx^i. \end{aligned}$$

Therefore, $\eta \circ df_x = 0$ iff $\sum_{j=1}^{d_2} \eta_j \partial f^j / \partial x^i = 0$ for every $i = 1, \dots, d_1$.

Example 2.9. — Let M, N be smooth manifolds where $\dim M \leq \dim N$, $f \in C^\infty(M, N)$ is a smooth embedding and let $S = f(M)$. Let us note that $N_f = (f_*(TM))^\perp$ is nothing but the conormal bundle of S . Thus, N_f can be seen as a generalization of the conormal bundle.

Hörmander proved the following [24, Thm 8.2.4]

PROPOSITION 2.10. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets. For any smooth map $f : \Omega_1 \rightarrow \Omega_2$ it is possible to extend the pull-back operation to the distributions $u \in \mathcal{D}'(\Omega_2)$ which satisfy $N_f \cap WF(u) = \emptyset$ in a unique way. Moreover the wave front set of f^*u is contained in the set $f^*WF(u) = \{(x; \eta \circ df_x) | (f(x); \eta) \in WF(u)\}$.*

We can give an equivalent criterion for which the pull-back theorem holds true:

LEMMA 2.11. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and Γ a closed cone in $\dot{T}^*\Omega_2$. For any smooth map $f : \Omega_1 \rightarrow \Omega_2$, $N_f \cap \Gamma = \emptyset$ is equivalent to $f^*\Gamma \cap \underline{0} = \emptyset$ where $\underline{0}$ denotes the zero section $\underline{0} \in T^*\Omega_1$.*

Proof. — We have the equality $N_f \cap \Gamma = \{(f(x); \eta) | (f(x); \eta) \in \Gamma \text{ and } \eta \circ df(x) = 0\}$ by definition of N_f . But we also have $f^*(\Gamma \cap N_f) = \{(x; \eta \circ df(x)) | (f(x); \eta) \in \Gamma \cap N_f\} = \{(x; 0) | (f(x); \eta) \in \Gamma \text{ and } \eta \circ df(x) = 0\} = f^*\Gamma \cap \underline{0}$. \square

Example 2.12. — If we go back to example 2.9, we can pull-back a distribution $u \in \mathcal{D}'(N)$ by an embedding f if $WF(u)$ does not meet the conormal of the embedded submanifold $S = f(M)$.

The sequential continuity of the pull-back [24, Thm 8.2.4] and Prop. 2.8 imply

PROPOSITION 2.13. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and Γ a closed cone in $\dot{T}^*\Omega_2$. For any smooth map $f : \Omega_1 \rightarrow \Omega_2$ such that $N_f \cap \Gamma = \emptyset$, the pull-back operation $f^* : \mathcal{D}'_\Gamma(\Omega_2) \rightarrow \mathcal{D}'_{f^*\Gamma}(\Omega_1)$, where $f^*\Gamma = \{(x; \eta \circ df_x) | (f(x); \eta) \in \Gamma\}$, is sequentially continuous and bounded.*

The proof of this theorem is sketched by Hörmander and Duistermaat [13, p. 19] and given in detail in [16, p. 155].

Note that, in the case of distributions on manifolds, the influence of orientation on the pull-back is often not correctly taken into account [35].

2.3.2. Push-forward

Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets. For any proper smooth map $f : \Omega_1 \rightarrow \Omega_2$ we define the *push-forward operation* $f_* : \mathcal{D}(\Omega_1) \rightarrow C^\infty(\Omega_2) \subset \mathcal{D}'(\Omega_2)$ by $\langle f_*g, h \rangle = \langle g, h \circ f \rangle = \langle g, f^*h \rangle$ [6, p. 527]. We can already note that the above definition means that the push-forward is the adjoint of the pull-back.

PROPOSITION 2.14. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets. For any smooth map $f : \Omega_1 \rightarrow \Omega_2$, the push-forward operation can be extended to any distribution $u \in \mathcal{D}'$ such that f is a proper map from $\text{supp } u$ to Ω_2 and $\text{WF}(f_*u) \subset \{(y; \eta) \in \dot{T}^*\Omega_2; \exists x \in \Omega_1 \text{ with } y = f(x) \text{ and } (x; \eta \circ df_x) \in \text{WF}(u) \cup \text{supp } u \times \{0\}\}$ [6, p. 528].*

We define the projections $\overline{\pi} : (x; \xi) \in T^*\mathbb{R}^{d_1} \mapsto \xi \in \mathbb{R}^{d_1}$ and $\underline{\pi} : (x; \xi) \in T^*\mathbb{R}^{d_1} \mapsto x \in \Omega$. Moreover,

PROPOSITION 2.15. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and Γ a closed cone in $\dot{T}^*\Omega_1$. For any smooth map $f : \Omega_1 \rightarrow \Omega_2$ and any closed subset C of Ω_1 such that $f|_C : C \rightarrow \Omega_2$ is proper and $\underline{\pi}(\Gamma) \subset C$, then f_* is sequentially continuous and bounded from $\{u \in \mathcal{D}'_\Gamma; \text{supp } u \subset C\}$ to $\mathcal{D}'_{f_*\Gamma}$, where $f_*\Gamma = \{(y; \eta) \in \dot{T}^*\Omega_2; \exists x \in \Omega_1 \text{ with } y = f(x) \text{ and } (x; \eta \circ df_x) \in \Gamma \cup \text{supp } u \times \{0\}\}$ [6, p. 528].*

2.3.3. Geometric interpretation.

Our goal is to give some geometric formalism which unifies the various transformations in cotangent space. To a map $f \in C^\infty(\Omega_1, \Omega_2)$, we associate a subset of the Cartesian product $T^*\Omega_1 \times T^*\Omega_2$ called **relation** which is defined as follows:

$$\text{Rel}(f) = \{((x; \xi), (y; \eta)) ; y = f(x), \xi = \eta \circ df, \eta \neq 0\} \subset T^*\Omega_1 \times T^*\Omega_2 \setminus (\underline{0} \times \underline{0}).$$

It is convenient to introduce the following notation

$$\text{Rel}(f)' = \{(x, y; \xi, -\eta) ; (x, y; \xi, \eta) \in \text{Rel}(f)\} \subset T^*\Omega_1 \times T^*\Omega_2$$

we also denote by $\omega_i, (i = 1, 2)$ the symplectic form of $T^*\Omega_i, (i = 1, 2)$. Then we give a symplectic interpretation of $\text{Rel}(f)$, $\text{Rel}(f)'$ is nothing

but the conormal $\{(x, f(x); -\eta \circ df, \eta); x \in \Omega_1, \eta \neq 0\}$ of the graph of f and is therefore a **Lagrangian submanifold** of the symplectic manifold $T^*\Omega_1 \times T^*\Omega_2$ with symplectic form $\omega_1 + \omega_2$ whereas $\text{Rel}(f)$ is Lagrangian for $\omega_1 - \omega_2$. Let Γ_2 (resp Γ_1) be a closed conic set in $\dot{T}^*\Omega_2$ (resp $\dot{T}^*\Omega_1$) and $f \in C^\infty(\Omega_1, \Omega_2)$, then we define the three sets $(N_f, f^*\Gamma_2, f_*\Gamma_1)$ in $(\dot{T}^*\Omega_2, T^*\Omega_1, \dot{T}^*\Omega_2)$ respectively in terms of the **relation** $\text{Rel}(f)$:

$$\begin{aligned} N_f &= \{(y; \eta) \in \dot{T}^*\Omega_2; ((x; 0), (y; \eta)) \in \text{Rel}(f)\} \\ f^*\Gamma_2 &= \{(x; \xi) \in T^*\Omega_1; ((x; \xi), (y; \eta)) \in \text{Rel}(f), (y; \eta) \in \Gamma_2\} \\ f_*\Gamma_1 &= \{(y; \eta) \in \dot{T}^*\Omega_2; ((x; \xi), (y; \eta)) \in \text{Rel}(f), (x; \xi) \in \Gamma_1 \cup \underline{0}\}. \end{aligned}$$

Note that the definition of $\text{Rel}(f)$ is coordinate free since the conormal of the graph of f is an intrinsic object, therefore the three above sets are intrinsically defined.

2.3.4. Tensor product

Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, $u \in \mathcal{D}'(\Omega_1)$, $v \in \mathcal{D}'(\Omega_2)$. The tensor product of u and v is the unique distribution $u \otimes v \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ such that for all $f \in \mathcal{D}(\Omega_1)$ and $g \in \mathcal{D}(\Omega_2)$, $\langle u \otimes v, f \otimes g \rangle = \langle u, f \rangle \langle v, g \rangle$ [34, p. 109]. The wave front set of the tensor product satisfies

$$\begin{aligned} \text{WF}(u \otimes v) \subset (\text{WF}(u) \times \text{WF}(v)) \quad \cup \quad ((\text{supp } u \times \{0\}) \times \text{WF}(v)) \\ \cup \quad (\text{WF}(u) \times (\text{supp } v \times \{0\})). \end{aligned}$$

The tensor product is then sequentially continuous in the following sense:

PROPOSITION 2.16. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$, $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and $\Gamma_1 \subset \dot{T}^*\Omega_1$, $\Gamma_2 \subset \dot{T}^*\Omega_2$ be two closed cones. Then, the mapping $(u, v) \mapsto u \otimes v$ is sequentially continuous and bounded from $\mathcal{D}'_{\Gamma_1}(\Omega_1) \times \mathcal{D}'_{\Gamma_2}(\Omega_2)$ to $\mathcal{D}'_{\Gamma}(\Omega_1 \times \Omega_2)$, where*

$$\Gamma = (\Gamma_1 \times \Gamma_2) \cup ((\Omega_1 \times \{0\}) \times \Gamma_2) \cup (\Gamma_1 \times (\Omega_2 \times \{0\})).$$

Proof. — The sequential continuity of the tensor product is well known [6, p. 511]. The boundedness follows from the hypocontinuity of the tensor product for the normal topology, which is proved in section 5. Indeed, if $f : E \times F \rightarrow G$ is a hypocontinuous bilinear map, then $f(A \times B)$ is bounded in G if A is bounded in E and B in F [3, p. III.31]. \square

2.3.5. Multiplication of distributions

The Hörmander famous product Theorem [24, p. 267] states that:

THEOREM 2.17. — *Let $\Omega \subset \mathbb{R}^d$ be an open set, u and v two distributions in $\mathcal{D}'(\Omega)$ such that $\text{WF}(u) \cap \text{WF}(v)' = \emptyset$, then the product of distributions uv is well defined in $\mathcal{D}'(\Omega)$ and*

$$(2.1) \quad \text{WF}(uv) \subset S_+ \cup S_u \cup S_v,$$

where $S_+ = \{(x; \xi + \eta) | (x; \xi) \in \text{WF}(u) \text{ and } (x; \eta) \in \text{WF}(v)\}$, $S_u = \{(x; \xi) | (x; \xi) \in \text{WF}(u) \text{ and } x \in \text{supp}(v)\}$ and $S_v = \{(x; \xi) | (x; \xi) \in \text{WF}(v) \text{ and } x \in \text{supp}(u)\}$.

Moreover [6, p. 526]

PROPOSITION 2.18. — *Let $\Omega \subset \mathbb{R}^n$ be an open set and Γ_1, Γ_2 be two closed cones in $\dot{T}^*\Omega$ such that $\Gamma_1 \cap \Gamma_2' = \emptyset$. Then the product of distributions is sequentially continuous and bounded from $\mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2}$ to \mathcal{D}'_{Γ} , where*

$$\Gamma = (\Gamma_1 +_{\Omega} \Gamma_2) \cup ((\Omega \times \{0\}) +_{\Omega} \Gamma_2) \cup (\Gamma_1 +_{\Omega} (\Omega \times \{0\})),$$

where, for any pair of cones Γ_1 and Γ_2 in $\dot{T}^*\Omega$, $\Gamma_1 +_{\Omega} \Gamma_2 = \{(x; \xi_1 + \xi_2) ; (x; \xi_1) \in \Gamma_1 \text{ and } (x; \xi_2) \in \Gamma_2\}$.

Proof. — This follows from the fact that the product of distributions is the composition of the tensor product and of the pull-back by the diagonal map $x \mapsto (x, x)$ [24, p. 267]. \square

3. Continuity for the Hörmander topology

A pseudo-topology is not always compatible with a topology [12] but Duistermaat [13, p. 18] noticed that Hörmander's pseudo-topology is compatible with a topology defined in terms of the following seminorms [19, p. 80]:

- (i) All the seminorms on $\mathcal{D}'(\mathbb{R}^n)$ for the weak topology: $\|u\|_{\phi} = |\langle u, \phi \rangle|$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$.
- (ii) The seminorms of the form

$$\|u\|_{N, V, \chi} = \sup_{k \in V} (1 + |k|)^N |\widehat{u\chi}(k)|,$$

where $N \geq 0$, $\chi \in \mathcal{D}(\mathbb{R}^n)$, and $V \in \mathbb{R}^n$ is a closed cone with $\text{supp } \chi \times V \cap \Gamma = \emptyset$.

These seminorms equip \mathcal{D}'_Γ with the structure of a locally convex space and the corresponding topology is usually called *Hörmander's topology*. It probably first appeared in the 1970-1971 lecture notes by Duistermaat [13], although the seminorms $\|\cdot\|_{N,V,\chi}$ are already mentioned by Hörmander [23, p. 128]. The topological convergence of sequences in *Hörmander's topology* is indeed equivalent to their convergence in the sense of Hörmander [9, p. 262] as in definition 2.1. We can now recall the statement of Proposition 2.7 and give the proof:

PROPOSITION 3.1. — *A sequence u_j in \mathcal{D}'_Γ converges to u in the sense of Hörmander iff it Mackey-converges to u .*

Proof. — It follows from the definition 2.1 and the definition of the Hörmander topology that a sequence converges in the sense of Hörmander iff it converges topologically for the Hörmander topology [36, p. 65].

Moreover, Lemma 21 in [9] states that a sequence of elements of \mathcal{D}'_Γ converges topologically iff it Mackey-converges. Thus, a sequence converges in the sense of Hörmander iff it Mackey-converges. \square

The functional properties of \mathcal{D}'_Γ with the Hörmander topology were studied in detail recently [9] and it was found that, although in $\mathcal{D}'(\Omega)$ every bounded map is continuous, this is no longer the case in \mathcal{D}'_Γ . Although we now know that the fundamental operations with distributions are bounded, we must investigate whether or not they are topologically continuous.

Duistermaat defines the Hörmander topology [13, p. 18] and claims without proof that the pull-back by a smooth map is continuous [13, p. 19]. However the following counterexample due to Alesker shows that this is not the case.

3.1. Alesker's counterexample

The following is a transcription of a letter sent to us by Semyon Alesker on 7 October 2013. The notation was slightly changed to make it consistent with the rest of the paper.

*I will give an example of a map $f : X \rightarrow Y$ and of a closed conic subset $\Gamma \subset T^*Y$ such that the pull-back map $f^* : \mathcal{D}_\Gamma(Y) \rightarrow \mathcal{D}_{f^*\Gamma}(X)$ is not topologically continuous for the Hörmander topology. We will show as a corollary that the tensor product is not topologically continuous (even separately).*

PROPOSITION 3.2. — *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the projection to the first coordinate. Let $\Gamma = \dot{T}^*\mathbb{R}$, so that $\mathcal{D}'_\Gamma(\mathbb{R}) = \mathcal{D}'(\mathbb{R})$, then $f^*\Gamma = \{(x_1, x_2; \xi_1, 0)\}$.*

We claim that the map $f^* : \mathcal{D}'_T(\mathbb{R}) \rightarrow \mathcal{D}'_{f^*\Gamma}(\mathbb{R}^2)$ is not topologically continuous for the Hörmander topology.

Proof. — Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi|_{[-1,1]} = 1$. Take $\chi = \varphi \otimes \varphi$, $V = \{(\xi_1, \xi_2) \in \mathbb{R}^2; |\xi_1| \leq |\xi_2|\}$ and $N = 0$. The intersection of V with $\{(\xi_1, 0); \xi_1 \neq 0\}$ is empty because $|\xi_1| \leq |\xi_2| = 0$ implies $\xi_1 = \xi_2 = 0$. Therefore, $\|\cdot\|_{N,V,\chi}$ is a seminorm of $\mathcal{D}'_{f^*\Gamma}$ and, if f^* were continuous, it would be possible to bound $\|f^*u\|_{N,V,\chi}$ with $\sup_i |\langle u, f_i \rangle|$ for a finite set of $f_i \in \mathcal{D}(\mathbb{R})$ and every $u \in \mathcal{D}'(\mathbb{R})$.

We are going to show that this is not the case. We have

$$\|f^*u\|_{0,V,\chi} = \sup_{\xi \in V} |\widehat{\varphi u}(\xi_1)| |\widehat{\varphi}(\xi_2)| = \sup_{\xi_1} |\widehat{\varphi u}(\xi_1)| \omega(\xi_1),$$

where $\omega(\xi_1) = \sup_{|\xi_2| \geq |\xi_1|} |\widehat{\varphi}(\xi_2)|$. It is clear that $\omega(\xi_1) > 0$ everywhere since $\widehat{\varphi}$ is a real analytic function. Thus we should show that the map $\mathcal{D}'(\mathbb{R}) \rightarrow \mathbb{R}$ given by $u \mapsto \sup_{\xi \in \mathbb{R}} |\widehat{\varphi u}(\xi)| \omega(\xi)$ is not continuous (for a fixed $\omega > 0$).

If the pull-back were continuous, there would be a finite set χ_1, \dots, χ_t of functions in $\mathcal{D}(\mathbb{R})$ such that

$$\|f^*u\|_{0,V,\chi} \leq \sup_{i=1,\dots,t} |\langle u, \chi_i \rangle|.$$

We can find ξ such that the functions χ_1, \dots, χ_t and $\varphi(x)e^{-ix\xi}$ are linearly independent. Then there exists $u \in \mathcal{D}'(\mathbb{R})$ such that $\langle u, \chi_i \rangle = 0$ for $i = 1, \dots, t$ and $\widehat{u\varphi}(\xi) = \langle u, \varphi e_\xi \rangle = 1 + 1/\omega(\xi)$, where $e_\xi(x) = e^{-ix\xi}$. Then, $\|f^*u\|_{0,V,\chi} = 1 + \omega(\xi)$ and we reach a contradiction. \square

Thus, the pull-back is not continuous. Moreover, the same example can be considered as the map $u \rightarrow u \otimes 1$. This shows that the tensor product is not separately continuous for the Hörmander topology.

SECOND PART

4. The normal topology

The Hörmander topology was chosen to be compatible with the convergence of sequences in the sense of Hörmander. But in fact, for any topology that is finer than the weak topology and coarser than the Mackey topology of \mathcal{D}'_T , a sequence is topologically convergent iff it is convergent in the sense of Hörmander [9]. Therefore, we can look for an alternative topology for which the fundamental operations have better continuity properties.

However, if we choose a topology that is too fine, then many maps f from a locally convex space E to \mathcal{D}'_Γ will not be continuous (because there are too many $f^{-1}(U)$ where U runs over the open sets of \mathcal{D}'_Γ) and, conversely, if the topology is too coarse, then many maps f from \mathcal{D}'_Γ to E will not be continuous. The choice of the proper topology is a subtle matter and we follow the advice given by Laurent Schwartz [33, p. 10] and define a *normal topology* on \mathcal{D}'_Γ .

DEFINITION 4.1. — *A Hausdorff locally convex space E is said to be a normal space of distributions if there are continuous injective linear maps $i : \mathcal{D}(\Omega) \hookrightarrow E$ and $j : E \hookrightarrow \mathcal{D}'(\Omega)$, where $\mathcal{D}'(\Omega)$ is equipped with its strong topology, such that: (i) The image of i is dense in E , (ii) for any f and g in $\mathcal{D}(\Omega)$, $\langle j \circ i(f), g \rangle = \int_\Omega f(x)g(x)dx$ [25, p. 319].*

The point of having $\mathcal{D}(\Omega) \hookrightarrow E \hookrightarrow \mathcal{D}'(\Omega)$ is that the dual E' can also be equipped with a normal topology and many interesting relations arise from that duality. In this paper, we choose to use *the coarsest topology* among all topologies defined in (4.1). In [9], it was shown that this topology is defined by the seminorms $\|\cdot\|_{N,V,\chi}$ of the Hörmander topology and the seminorms p_B of the strong topology of $\mathcal{D}'(\Omega)$: $p_B(u) = \sup_{f \in B} |\langle u, f \rangle|$, where B runs over the bounded sets of $\mathcal{D}(\Omega)$ [9]. This topology will be simply called **the normal topology** on \mathcal{D}'_Γ in the following. From now on, the space of distributions $\mathcal{D}'(\Omega)$ will always be equipped with the strong topology.

With the normal topology, it will be shown that the pull-back and push-forward are continuous and that the tensor product and distribution product are hypocontinuous. We cannot expect the distribution product to be (jointly) continuous because, in the case of $\Gamma_1 = \emptyset$ and $\Gamma_2 = \dot{T}^*(\Omega_2)$, appendix 10.3 states that $D'_{\Gamma_1} = C^\infty(\Omega)$ and $D'_{\Gamma_2} = \mathcal{D}'(\Omega)$ and it is known that the product $C^\infty(\Omega) \times \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)$ is hypocontinuous [34, p. 119] but not (jointly) continuous [28, 29].

Let us recall the definition of hypocontinuity which, for many purposes, is as good as continuity [34, p. 104]:

DEFINITION 4.2. — [36, p. 423] *Let E , F and G be topological vector spaces. A bilinear map $f : E \times F \rightarrow G$ is said to be hypocontinuous if: (i) for every neighborhood W of zero in G and every bounded set $A \subset E$ there is a neighborhood V of zero in F such that $f(A \times V) \subset W$ and (ii) for every neighborhood W of zero in G and every bounded set $B \subset F$ there is a neighborhood U of zero in E such that $f(U \times B) \subset W$.*

If E , F and G are locally convex spaces with topologies defined by the families of seminorms $(p_i)_{i \in I}$, $(q_j)_{j \in J}$ and $(r_k)_{k \in K}$, respectively, the definition of hypocontinuity can be translated into the following two conditions: (i) For every bounded set A of E and every seminorm r_k , there is a constant M and a finite set of seminorms q_{j_1}, \dots, q_{j_n} (both depending only on k and A) such that

$$(4.1) \quad \forall x \in A, r_k(f(x, y)) \leq M \sup\{q_{j_1}(y), \dots, q_{j_n}(y)\};$$

and (ii) For every bounded set B of F and every seminorm r_k , there is a constant M and a finite set of seminorms p_{i_1}, \dots, p_{i_n} (both depending only on k and B) such that

$$(4.2) \quad \forall y \in B, r_k(f(x, y)) \leq M \sup\{p_{i_1}(x), \dots, p_{i_n}(x)\}.$$

Equivalently [26, p. 155], for every bounded set A of E and every bounded set B of F the sets of maps $\{f_x; x \in A\}$ and $\{f_y; y \in B\}$ are equicontinuous, where $f_x : E \rightarrow G$ and $f_y : F \rightarrow G$ are defined by $f_x(y) = f_y(x) = f(x, y)$.

Important properties of hypocontinuous maps are given in

PROPOSITION 4.3. — [25, p. 359] *Let E , F and G be topological vector spaces and $f : E \times F \rightarrow G$ a hypocontinuous map. Then, for every bounded set A of E and B of F : (i) f is continuous on $A \times F$ and $E \times B$; (ii) f is uniformly continuous on $A \times B$; (iii) $f(A \times B)$ is bounded in G .*

LEMMA 4.4. — *Let Ω be an open set of \mathbb{R}^d and B a bounded set in $\mathcal{D}'(\Omega)$, then for every $\chi \in \mathcal{D}(\Omega)$ there exist an integer M and a constant C (both depending only on B and on an arbitrary relatively compact open neighborhood of $\text{supp } \chi$) such that*

$$\sup_{u \in B} \sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-M} |\widehat{u\chi}(\xi)| < 2^M C \text{Vol}(K) \pi_{M,K}(\chi),$$

where $K = \text{supp } \chi$.

Proof. — Let Ω_0 be a relatively compact open neighborhood of $K = \text{supp } \chi$. According to Schwartz [34, p. 86], for any bounded set B in $\mathcal{D}'(\Omega)$, there is an integer M (depending only on B and Ω_0) such that every $u \in B$ can be expressed in Ω_0 as $u = \partial^\alpha f_u$ for $|\alpha| \leq M$, where f_u is a continuous function. Moreover, there is a constant C (depending only on B and Ω_0) such that $|f_u(x)| \leq C$ for all $x \in \Omega_0$ and $u \in B$. Thus,

$$\begin{aligned} \widehat{u\chi}(\xi) &= \int_{\Omega_0} e^{-i\xi \cdot x} \chi(x) \partial^\alpha f_u(x) dx = (-1)^{|\alpha|} \int_{\Omega_0} f_u(x) \partial^\alpha (e^{-i\xi \cdot x} \chi(x)) dx \\ &= (-1)^{|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} (i\xi)^\beta \int_{\Omega_0} f_u(x) e^{-i\xi \cdot x} \partial^{\beta-\alpha} \chi(x) dx. \end{aligned}$$

By using $|(i\xi)^\beta| \leq (1 + |k|)^M$ if $|\beta| \leq M$ we obtain

$$\begin{aligned} (1 + |\xi|)^{-M} |\widehat{u\chi}(\xi)| &\leq \sup_{|\alpha| \leq M} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left| \int_{\Omega_0} f_u(x) e^{-i\xi \cdot x} \partial^{\beta-\alpha} \chi(x) dx \right| \\ &\leq 2^M C \text{Vol}(K) \pi_{M,K}(\chi). \end{aligned}$$

□

5. Tensor product of distributions.

Let Ω_1 and Ω_2 be open sets in \mathbb{R}^{d_1} and \mathbb{R}^{d_2} , respectively, and $(u, v) \in \mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2}$, where $\mathcal{D}'_{\Gamma_1} \subset \mathcal{D}'(\Omega_1)$ and $\mathcal{D}'_{\Gamma_2} \subset \mathcal{D}'(\Omega_2)$. Then the tensor product $u \otimes v$ belongs to $\mathcal{D}'_{\Gamma} \subset \mathcal{D}'(\Omega_1 \times \Omega_2)$ where

$$\begin{aligned} \Gamma &= (\Gamma_1 \times \Gamma_2) \cup ((\Omega_1 \times \{0\}) \times \Gamma_2) \cup (\Gamma_1 \times (\Omega_2 \times \{0\})) \\ &= (\Gamma_1 \cup \{\underline{0}\}_1) \times (\Gamma_2 \cup \{\underline{0}\}_2) \setminus \{(\underline{0}, \underline{0})\}, \end{aligned}$$

where $\{\underline{0}\}_1$ means $\Omega_1 \times \{0\}$, $\{\underline{0}\}_2$ means $\Omega_2 \times \{0\}$ and $\{\underline{0}, \underline{0}\}$ means $(\Omega_1 \times \Omega_2) \times \{0, 0\}$. Our goal in this section is to show that the tensor product is hypocontinuous for the normal topology. We denote by $(z; \zeta)$ the coordinates in $T^*(\Omega_1 \times \Omega_2)$, where $z = (x, y)$ with $x \in \Omega_1$ and $y \in \Omega_2$, $\zeta = (\xi, \eta)$ with $\xi \in \mathbb{R}^{d_1}$ and $\eta \in \mathbb{R}^{d_2}$. We also denote $d = d_1 + d_2$, so that $\zeta \in \mathbb{R}^d$.

LEMMA 5.1. — *The seminorms of the strong topology of \mathcal{D}'_{Γ} and the family of seminorms:*

$$(5.1) \quad \|t_1 \otimes t_2\|_{N, V, \varphi_1 \otimes \varphi_2} = \sup_{\zeta \in V} (1 + |\zeta|)^N |\widehat{t_1 \varphi_1}(\xi)| |\widehat{t_2 \varphi_2}(\eta)|,$$

where $\zeta = (\xi, \eta)$, $(\varphi_1, \varphi_2) \in \mathcal{D}(\Omega_1) \times \mathcal{D}(\Omega_2)$ and $V \subset \mathbb{R}^d$, are such that $(\text{supp } (\varphi_1 \otimes \varphi_2) \times V) \cap \Gamma = \emptyset$, are a fundamental system of seminorms for the normal topology of \mathcal{D}'_{Γ} .

Proof. —

We use the following lemma [19, p. 80]

LEMMA 5.2. — *Let Ω be an open set in \mathbb{R}^n . If we have a family, indexed by $\alpha \in A$, of $\chi_\alpha \in \mathcal{D}(\Omega)$ and of closed cones $V_\alpha \subset (\mathbb{R}^n \setminus \{0\})$ such that $(\text{supp } \chi_\alpha \times V_\alpha) \cap \Gamma = \emptyset$ and*

$$\Gamma^c = \bigcup_{\alpha \in A} \{(x, \xi) \in \dot{T}^*\Omega; \chi_\alpha(x) \neq 0, \xi \in V_\alpha^\circ\},$$

then the topology of \mathcal{D}'_{Γ} is already defined by the topology of $\mathcal{D}'(\Omega)$ and the seminorms $\|\cdot\|_{N, V_\alpha, \chi_\alpha}$.

It is clear that the family indexed by $\varphi_1 \otimes \varphi_2$ and V such that $\text{supp}(\varphi_1 \otimes \varphi_2) \times V \cap \Gamma = \emptyset$ satisfies the hypothesis of the lemma. \square

To establish the hypocontinuity of the tensor product, we consider an arbitrary bounded set $B \subset \mathcal{D}'_{\Gamma_1}(\Omega_1)$ and, according to eq. (4.1), we must show that, for every seminorm r_k of $\mathcal{D}'_{\Gamma}(\Omega_1 \times \Omega_2)$, there is a constant M and a finite number of seminorms q_j such that $r_k(u \otimes v) \leq M \sup_j q_j(v)$ for every $u \in B$ and every $v \in \mathcal{D}'_{\Gamma_2}(\Omega_2)$. By Schwartz' theorem [34, p. 110] we already know that this is true for every seminorm r_k of the strong topology of $\mathcal{D}'_{\Gamma}(\Omega_1 \times \Omega_2)$. It remains to show it for every $\|\cdot\|_{N,V,\varphi_1 \otimes \varphi_2}$. This will be done by first defining a suitable partition of unity on $\Omega_1 \times \Omega_2$ and its corresponding cones. Then, this partition of unity will be used to bound the seminorms by standard microlocal techniques.

LEMMA 5.3. — *Let Γ_1, Γ_2 be closed cones in $\dot{T}^*\Omega_1$ and $\dot{T}^*\Omega_2$, respectively. Set $\Gamma = (\Gamma_1 \cup \{\underline{0}\}) \times (\Gamma_2 \cup \{\underline{0}\}) \setminus \{(\underline{0}, \underline{0})\} \subset \dot{T}^*\mathbb{R}^d$ then for all closed cones $V \subset \mathbb{R}^d$ and $\chi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ such that $(\text{supp } \chi \times V) \cap \Gamma = \emptyset$, there exist a partition of unity $(\psi_{j1} \otimes \psi_{j2})_{j \in J}$ of $\Omega_1 \times \Omega_2$, which is finite on $\text{supp } \chi$, and a family of closed cones $(W_{j1} \times W_{j2})_{j \in J}$ in $(\mathbb{R}^{d_1} \setminus \{0\}) \times (\mathbb{R}^{d_2} \setminus \{0\})$ such that*

$$(5.2) \quad (\text{supp } \psi_{j1} \times W_{j1}^c) \cap \Gamma_1 = (\text{supp } \psi_{j2} \times W_{j2}^c) \cap \Gamma_2 = \emptyset,$$

$$(5.3) \quad V \cap ((W_{j1} \cup \{0\}) \times (W_{j2} \cup \{0\})) = \emptyset,$$

$$\text{if } \text{supp } \chi \cap \text{supp } (\psi_{j1} \otimes \psi_{j2}) \neq \emptyset.$$

Proof. — We first set some useful notations and observations. For any $D \in \mathbb{N}$, on the base of the identification $T^*\mathbb{R}^D \simeq \mathbb{R}^D \oplus (\mathbb{R}^D)^*$, we denote by $\underline{\pi} : T^*\mathbb{R}^D \rightarrow \mathbb{R}^D$ the projection on the first factor and by $\bar{\pi} : T^*\mathbb{R}^D \rightarrow (\mathbb{R}^D)^*$ the projection on the second factor. We use the distance d_∞ on \mathbb{R}^D (or $(\mathbb{R}^D)^*$) defined by $d_\infty(u, v) := \sup_{1 \leq i \leq D} |u^i - v^i|$. For $u \in \mathbb{R}^D$ and $r \geq 0$ we then set $\overline{B}(u, r) = \{v \in \mathbb{R}^D; d_\infty(u, v) \leq r\}$ and, for any subset $Q \subset \mathbb{R}^D$, $Q_{,r} := \{v \in \mathbb{R}^D; d_\infty(v, Q) \leq r\}$. We note the useful property that, for any pair of sets $Q_1 \subset \mathbb{R}^{d_1}$ and $Q_2 \subset \mathbb{R}^{d_2}$, $(Q_1 \times Q_2)_{,r} = Q_{1,r} \times Q_{2,r}$ (in particular, if $(x, y) \in \Omega_1 \times \Omega_2$, $\overline{B}((x, y), r) = \overline{B}(x, r) \times \overline{B}(y, r)$). Lastly for any closed conic subset $W \subset (\mathbb{R}^D)^* \setminus \{0\}$, we set $\overline{W} := W \cup \{0\}$ for short and $UW := S^{D-1} \cap W$. Similarly if Γ is a conic subset of $T^*\mathbb{R}^D$, we set $U\Gamma = (\mathbb{R}^D \times S^{D-1}) \cap \Gamma$ and $\overline{\Gamma} = \Gamma \cup \underline{0} \subset T^*\mathbb{R}^D$ where $\underline{0}$ is the zero section of $T^*\mathbb{R}^D$.

We will prove that there exists a family of open ball $(B_{j1} \times B_{j2})_{j \in J}$ which covers $\Omega_1 \cap \Omega_2$, which is finite over any compact subset of $\Omega_1 \times \Omega_2$ and in particular on $\text{supp } \chi$ and such that $(\overline{B_{j1}} \times W_{j1}^c) \cap \Gamma_1 = (\overline{B_{j2}} \times W_{j2}^c) \cap \Gamma_2 = \emptyset$ and that $V \cap (\overline{W_{j1}} \times \overline{W_{j2}}) = \emptyset$, if $\text{supp } \chi \cap (\overline{B_{j1}} \times \overline{B_{j2}}) \neq \emptyset$. The

conclusion of the lemma will then follow by constructing a partition of unity $(\psi_{j1} \otimes \psi_{j2})_{j \in J}$ such that $\text{supp } \psi_{j1} = \overline{B}_{j1}$ and $\text{supp } \psi_{j2} = \overline{B}_{j2}$, $\forall j \in J$, by using standard arguments.

Step 1 — Hypothesis $(\text{supp } \chi \times V) \cap \Gamma = \emptyset$ implies that there exists some $\delta > 0$ such that $d_\infty(\text{supp } \chi \times UV, U\Gamma) \geq 4\delta$. Consider $K := (\text{supp } \chi)_{,\delta}$ we note then that $d_\infty(K \times UV, U\Gamma) \geq 3\delta$. W.l.g. we can assume that δ has been chosen so that $K \subset \Omega_1 \times \Omega_2$. Obviously $\Omega_1 \times \Omega_2$ is covered by $(B((x, y), \delta))_{(x, y) \in \Omega_1 \times \Omega_2}$. Moreover all balls $B((x, y), \delta)$ are contained in K if $(x, y) \in \text{supp } \chi$ and $\text{supp } \chi$ is covered by the subfamily $(B((x, y), \delta))_{(x, y) \in \text{supp } \chi}$. Since $\text{supp } \chi$ is compact we can thus extract a countable family of balls $(B_i)_{i \in I} = (B_{i1} \times B_{i2})_{i \in I}$ which covers $\Omega_1 \times \Omega_2$ and which is finite over $\text{supp } \chi$.

We now set $\gamma := \pi(\pi^{-1}(K) \cap \Gamma)$ and $U\gamma := \pi(\pi^{-1}(K) \cap U\Gamma)$ and we estimate the distance of $U\gamma$ to UV :

$$\begin{aligned} d_\infty[U\gamma, UV] &= \inf_{\xi \in \pi(\pi^{-1}(K) \cap U\Gamma)} \inf_{\eta \in UV} d_\infty(\xi, \eta) \\ &= \inf_{(u, \xi) \in U\Gamma; u \in K} \inf_{(v, \eta) \in K \times UV} d_\infty(\xi, \eta) \\ &= \inf_{(u, \xi) \in U\Gamma; u \in K} \inf_{(v, \eta) \in K \times UV} d_\infty((u, \xi), (v, \eta)), \end{aligned}$$

where the last equality is due to the fact that one can choose $v = u$ in the minimization. We deduce that, by removing the constraint $u \in K$ in the minimization,

$$\begin{aligned} d_\infty[U\gamma, UV] &\geq \inf_{(u, \xi) \in U\Gamma} \inf_{(v, \eta) \in K \times UV} d_\infty((u, \xi), (v, \eta)) \\ &= d_\infty(K \times UV, U\Gamma) \geq 3\delta. \end{aligned}$$

Step 2 — Since γ and V are cones, we deduce from the previous inequality that

$$\forall \xi \in \gamma, \quad d_\infty(\xi, V) \geq 2\|\xi\|\delta.$$

For any $i \in I$ such that the ball B_i is centered at a point in $\text{supp } \chi$, the inclusion $\overline{B}_i \subset K$ implies $\pi(\pi^{-1}(\overline{B}_i) \cap \Gamma) \subset \gamma$. We hence have also

$$(5.4) \quad \forall \xi \in \pi(\pi^{-1}(\overline{B}_i) \cap \Gamma) \quad d_\infty(\xi, V) \geq 2\|\xi\|\delta.$$

We now set $\overline{W}_{i1} := \{\xi_1 \in (\mathbb{R}^{d_1})^*; d_\infty(\xi_1, \pi(\pi^{-1}(\overline{B}_{i1}) \cap \Gamma_1)) \leq \|\xi_1\|\delta\}$, $\overline{W}_{i2} := \{\xi_2 \in (\mathbb{R}^{d_2})^*; d_\infty(\xi_2, \pi(\pi^{-1}(\overline{B}_{i2}) \cap \Gamma_2)) \leq \|\xi_2\|\delta\}$ and $W_{i1} := \overline{W}_{i1} \setminus \{0\}$, $W_{i2} := \overline{W}_{i2} \setminus \{0\}$. By the definition of W_{i1} , $W_{i1}^c \cap \pi(\pi^{-1}(\overline{B}_{i1}) \cap \Gamma_1) = \emptyset$, which is equivalent to $(\overline{B}_{i1} \times W_{i1}^c) \cap \Gamma_1 = \emptyset$. Similarly $(\overline{B}_{i2} \times W_{i2}^c) \cap \Gamma_2 = \emptyset$.

On the other hand, since

$$\begin{aligned} \pi(\pi^{-1}(\overline{B}_{i1}) \cap \Gamma_1) \times \pi(\pi^{-1}(\overline{B}_{i2}) \cap \Gamma_2) &= \pi[\pi^{-1}(\overline{B}_{i1} \times \overline{B}_{i2}) \cap (\overline{\Gamma}_1 \times \overline{\Gamma}_2)] \\ &= \pi[\pi^{-1}(\overline{B}_i) \cap \overline{\Gamma}], \quad \overline{B}_i = \overline{B}_{i1} \times \overline{B}_{i2} \end{aligned}$$

because

$$\begin{aligned} & \{\xi_1; \exists(x_1; \xi_1) \in \overline{\Gamma_1}, x_1 \in \overline{B_{i1}}\} \times \{\xi_2; \exists(x_2; \xi_2) \in \overline{\Gamma_2}, x_2 \in \overline{B_{i2}}\} \\ &= \{(\xi_1, \xi_2); \exists(x_1, x_2; \xi_1, \xi_2) \in \overline{\Gamma_1} \times \overline{\Gamma_2}, (x_1, x_2) \in \overline{B_{i1}} \times \overline{B_{i2}}\} \end{aligned}$$

we also have

$$\begin{aligned} \overline{W_{i1}} \times \overline{W_{i2}} &= \{(\xi_1, \xi_2) \in (\mathbb{R}^d)^*; d_\infty[(\xi_1, \xi_2), \pi(\pi^{-1}(\overline{B_i}) \cap \Gamma)] \\ &\leq \sup(\|\xi_1\|, \|\xi_2\|)\delta\}. \end{aligned}$$

Hence by (5.4), we deduce that $\overline{W_{i1}} \times \overline{W_{i2}}$ does not meet V . \square

In the remaining part of the paper, we may identify abusively \mathbb{R}^d and $(\mathbb{R}^d)^*$.

We also introduce the notation $e_\zeta(x, y) = e^{i(\xi \cdot x + \eta \cdot y)}$ where $\zeta = (\xi, \eta)$.

To estimate $\|u \otimes v\|_{N, V, \varphi_1 \otimes \varphi_2}$, we use Lemma 5.3 to find a partition of unity $(\psi_{j1} \otimes \psi_{j2})_{j \in J}$ which is finite on $\text{supp}(\varphi_1 \otimes \varphi_2)$ to write

$$\begin{aligned} \widehat{u\varphi_1}(\xi) \widehat{v\varphi_2}(\eta) &= \mathcal{F}(u\varphi_1 \otimes v\varphi_2)(\zeta) = \langle u \otimes v, (\varphi_1 \otimes \varphi_2)e_\zeta \rangle \\ &= \sum_j \langle u \otimes v, (\varphi_1 \psi_{j1} \otimes \varphi_2 \psi_{j2})e_\zeta \rangle = \sum_j \widehat{u\varphi_1 \psi_{j1}}(\xi) \widehat{v\varphi_2 \psi_{j2}}(\eta). \end{aligned}$$

Therefore $\|u \otimes v\|_{N, V, \varphi_1 \otimes \varphi_2} \leq \sum_j \|u \otimes v\|_{N, V, \varphi_1 \psi_{j1} \otimes \varphi_2 \psi_{j2}}$, where the sum over j is finite. Each seminorm on the right hand side is bounded by the following lemma.

LEMMA 5.4. — *Let Γ_1, Γ_2 and Γ be closed cones as in the previous lemma, $\psi_1 \in \mathcal{D}(\Omega_1)$ $\psi_2 \in \mathcal{D}(\Omega_2)$ such that $(\text{supp}(\psi_1 \otimes \psi_2) \times V) \cap \Gamma = \emptyset$ and closed cones W_1 and W_2 in $\mathbb{R}^{d_1} \setminus \{0\}$ and $\mathbb{R}^{d_2} \setminus \{0\}$ such that*

$$(5.5) \quad (W_1 \cup \{0\}) \times (W_2 \cup \{0\}) \cap V = \emptyset,$$

$$(5.6) \quad (\text{supp} \psi_k \times W_k^c) \cap \Gamma_k = \emptyset, \text{ for } k = 1, 2.$$

Then, for every bounded set $A \subset \mathcal{D}'_{\Gamma_1}$ and every integer N , there are constants m, M_1, M_2 and a bounded set $B \subset \mathcal{D}(K)$, where K is an arbitrary compact neighborhood of $\text{supp} \psi_2$, such that

$$\|t_1 \otimes t_2\|_{N, V, \psi_1 \otimes \psi_2} \leq M_1 \|t_2\|_{N, C_\beta, \psi_2} + M_2 \|t_2\|_{N+m, C_\beta, \psi_2} + p_B(t_2),$$

for every $t_1 \in A$ and $t_2 \in \mathcal{D}'_{\Gamma_2}$, where C_β is an arbitrary compact neighborhood of W_2 .

Proof. — We want to calculate

$$\|t_1 \otimes t_2\|_{N, V, \psi_1 \otimes \psi_2} = \sup_{\zeta \in V} (1 + |\zeta|)^N |\mathcal{F}(t_1 \psi_1 \otimes t_2 \psi_2)(\zeta)|.$$

We denote $u = t_1 \psi_1$, $v = t_2 \psi_2$ and $I = \widehat{u \otimes v}$. From $e_{(\xi, \eta)} = e_\xi \otimes e_\eta$ we find that $I(\xi, \eta) = \langle t, e_{(\xi, \eta)} \rangle = \langle u \otimes v, e_\xi \otimes e_\eta \rangle = \langle u, e_\xi \rangle \langle v, e_\eta \rangle = \widehat{u}(\xi) \widehat{v}(\eta)$.

By the shrinking lemma we can slightly enlarge W_1 and W_2 to closed cones having the same properties. Thus, there are two homogeneous functions of degree zero α and β on \mathbb{R}^{d_1} and \mathbb{R}^{d_2} , respectively, which are smooth except at the origin, non-negative and bounded by 1, such that: (i) $\alpha|_{W_1 \cup \{0\}} = 1$ and $\beta|_{W_2 \cup \{0\}} = 1$; (ii) $(\text{supp } \alpha \times \text{supp } \beta) \cap V = \emptyset$; (iii) $(\text{supp } \psi_1 \times \text{supp } (1 - \alpha)) \cap \Gamma_1 = \emptyset$; (iv) $(\text{supp } \psi_2 \times \text{supp } (1 - \beta)) \cap \Gamma_2 = \emptyset$. We can write $I = I_1 + I_2 + I_3 + I_4$ where (recalling that $\zeta = (\xi, \eta)$)

$$\begin{aligned} I_1(\zeta) &= \alpha(\xi)\widehat{u}(\xi)\beta(\eta)\widehat{v}(\eta), \\ I_2(\zeta) &= \alpha(\xi)\widehat{u}(\xi)(1 - \beta)(\eta)\widehat{v}(\eta), \\ I_3(\zeta) &= (1 - \alpha)(\xi)\widehat{u}(\xi)\beta(\eta)\widehat{v}(\eta), \\ I_4(\zeta) &= (1 - \alpha)(\xi)\widehat{u}(\xi)(1 - \beta)(\eta)\widehat{v}(\eta). \end{aligned}$$

The term $I_1(\zeta) = 0$ because, by condition (ii) $\alpha(\xi)\beta(\eta) = 0$ for $(\xi, \eta) \in V$. Condition (iii) implies that

$$|(1 - \alpha)(\xi)\widehat{u}(\xi)| \leq \sup_{\xi \in C_\alpha} |\widehat{t_1\psi_1}(\xi)| \leq (1 + |\xi|)^{-N} \|t_1\|_{N, C_\alpha, \psi_1},$$

where $\xi \in C_\alpha = \text{supp } (1 - \alpha)$. This gives us immediately, with $C_\beta = \text{supp } (1 - \beta)$

$$\begin{aligned} |I_4(\zeta)| &\leq (1 + |\xi|)^{-N} (1 + |\eta|)^{-N} \|t_1\|_{N, C_\alpha, \psi_1} \|t_2\|_{N, C_\beta, \psi_2} \\ &\leq (1 + |\zeta|)^{-N} \|t_1\|_{N, C_\alpha, \psi_1} \|t_2\|_{N, C_\beta, \psi_2}, \end{aligned}$$

because $1 + |(\xi, \eta)| \leq 1 + |\xi| + |\eta| \leq (1 + |\xi|)(1 + |\eta|)$. Since the set A is bounded in \mathcal{D}'_{Γ_1} there is a constant $M_1 = \sup_{t_1 \in A} \|t_1\|_{N, C_\alpha, \psi_1}$ such that

$$|I_4(\zeta)| \leq (1 + |\zeta|)^{-N} M_1 \|t_2\|_{N, C_\beta, \psi_2}.$$

To estimate I_2 , we use the fact that, $u = t_1\psi_1$ being a compactly supported distribution there is an integer m such that, for all $t_1 \in A$,

$$|\alpha(\xi)\widehat{u}(\xi)| \leq |\widehat{u}(\xi)| \leq (1 + |\xi|)^m \|\theta^{-m}\widehat{u}\|_{L^\infty}.$$

Moreover, as for the estimate of I_4 , we have $|(1 - \beta)(\eta)\widehat{v}(\eta)| \leq (1 + |\eta|)^{-N-m} \|t_2\|_{N+m, C_\beta, \psi_2}$. The set $\{\zeta \in \text{supp } \alpha \times C_\beta; |\zeta| = 1\} \cap V$ is **compact** and avoids the set of all elements of the form $\zeta = (\xi, 0), \xi \in \text{supp } \alpha \setminus \{0\}$. Otherwise, we would find some sequence $(\xi_n, \eta_n) \rightarrow (\xi, 0) \in ((\text{supp } \alpha \times \{0\}) \cap V) \subset ((\text{supp } \alpha \times \text{supp } \beta) \cap V)$ which contradicts the condition (ii). Let $\epsilon > 0$ be the smallest value of $|\eta|$ in this set. Then, the functions α and β being homogeneous of degree zero, $\text{supp } \alpha \times C_\beta \cap V$ is a cone in \mathbb{R}^d and $|\eta|/|\zeta| \geq \epsilon$ for all $\zeta = (\xi, \eta)$ in the set $\text{supp } \alpha \times C_\beta \cap V$. Thus, $(1 + |\eta|)^{-N-m} \leq \epsilon^{-N-m} (1 + |(\xi, \eta)|)^{-N-m}$ and

$$|I_2(\zeta)| \leq \|\theta^{-m}\widehat{t_1\psi_1}\|_{L^\infty} \|t_2\|_{N+m, C_\beta, \psi_2} \epsilon^{-N-m} (1 + |\zeta|)^{-N},$$

for every $\zeta \in V$, because $|\xi| \leq |(\eta, \xi)|$. By lemma 4.4, $\|\theta^{-m} \widehat{t_1 \psi_1}\|_{L^\infty}$ is uniformly bounded for $t_1 \in A$ by a constant $M'_2 = \sup_{t_1 \in A} \|\theta^{-m} \widehat{t_1 \psi_1}\|_{L^\infty}$, $\theta = 1 + |\xi|$. Therefore, there is a constant $M_2 = M'_2 \epsilon^{-N-m}$ such that, for every $t_1 \in A$ and every $t_2 \in \mathcal{D}'_{\Gamma_2}$, $|I_2(\zeta)| \leq M_2 \|t_2\|_{N+m, C_\beta, \psi_2}$.

The term I_3 is treated differently because we want to get the following result: for every bounded set A in \mathcal{D}'_{Γ_1} and every seminorm $\|\cdot\|_{N, V, \chi}$, there is a bounded set $B \in \mathcal{D}(\Omega_2)$ such that $\forall \zeta \in V$, $I_3(\zeta) \leq p_B(t_2)(1 + |\zeta|)^{-N}$ for every $t_2 \in \mathcal{D}'_{\Gamma_2}$. This special form of eq. (4.1) is possible because the union of bounded sets is a bounded set and the multiplication of a bounded set by a positive constant M is a bounded set.

Therefore, we write $I_3(\zeta) = \langle t_2, f_\zeta \rangle$, where

$$f_{(\xi, \eta)}(y) = (1 - \alpha)(\xi) \widehat{u}(\xi) \beta(\eta) \psi_2(y) e_\eta(y)$$

and we must show that the set $B = \{(1 + |\zeta|)^N f_\zeta; \zeta \in V\}$ is a bounded set of $\mathcal{D}(\Omega_2)$. A subset B of $\mathcal{D}(\Omega_2)$ is bounded iff there is a compact set K and a constant M_n for every integer n such that $\text{supp } f \subset K$ and $\pi_{n, K}(f) \leq M_n$ for every $f \in B$. All f_ζ are supported on $\text{supp } \psi_2$ and are smooth functions because ψ_2 and e_η are smooth. We have to prove that, if t_1 runs over a bounded set of \mathcal{D}'_{Γ_1} , then there are constants M_n such that $\pi_{n, K}(f_\zeta) \leq M_n$ for all $\zeta \in V$, where K is a compact neighborhood of $\text{supp } \psi_2$. We start from

$$\pi_{n, K}(f_{(\xi, \eta)}) = |(1 - \alpha)(\xi) \widehat{u}(\xi) \beta(\eta)| \pi_{n, K}(\psi_2 e_\eta).$$

We notice that $\pi_{n, K}(\psi_2 e_\eta) \leq 2^n \pi_{n, K}(\psi_2) \pi_{n, K}(e_\eta)$ and that $\pi_{n, K}(e_\eta) \leq |\eta|^n$. As for the estimate of I_2 , we have $|(1 - \alpha)(\xi) \widehat{u}(\xi)| \leq (1 + |\xi|)^{-N-n} \|t_1\|_{N+n, C_\alpha, \psi_1}$ because $(\text{supp } \varphi_1 \times \text{supp } (1 - \alpha)) \cap \Gamma_1 = \emptyset$ and $(1 + |\xi|)^{-N-n} \leq \epsilon^{-N-n} (1 + |(\xi, \eta)|)^{-N-n}$ for some ϵ because $(\xi, \eta) \in (V \cap \text{supp } (1 - \alpha) \times \text{supp } \beta)$. Therefore

$$\pi_{n, K}(f_\zeta) \leq \|t_1\|_{N+n, C_\alpha, \psi_1} 2^n \pi_{n, K}(\psi_2) \epsilon^{-N-n} (1 + |\zeta|)^{-N},$$

because $|\eta|^n (1 + |(\xi, \eta)|)^{-n} \leq 1$. If t_1 belongs to a bounded set A of \mathcal{D}'_{Γ_1} , then for each N $\|t_1\|_{N, C_\alpha, \psi_1}$ is uniformly bounded. The estimate of I_3 is finally

$$|I_3(\zeta)| \leq p_B(t_2) (1 + |\zeta|)^{-N}.$$

□

For each j , the conditions of the lemma hold if we put $\psi_1 = \varphi_1 \psi_{j1}$, $\psi_2 = \varphi_2 \psi_{j2}$, $W_1 = W_{j1}$ and $W_2 = W_{j2}$. Thus, for every bounded set A in

\mathcal{D}'_{Γ_1} , every $u \in A$ and every $v \in \mathcal{D}'_{\Gamma_2}$ we have

$$\begin{aligned} \|u \otimes v\|_{N, V, \varphi_1 \otimes \varphi_2} &\leq \sum_j \|u \otimes v\|_{N, V, \varphi_1 \psi_{j1} \otimes \varphi_2 \psi_{j2}} \\ &\leq \sum_j M_{1j} \|v\|_{N, C_{\beta_j}, \varphi_2 \psi_{j2}} + M_2 \|v\|_{N+m, C_{\beta_j}, \varphi_2 \psi_{j2}} + p_{B_j}(v). \end{aligned}$$

Since the sum over j is finite, this means that the family of maps $u \times v \mapsto u \otimes v$, where $u \in A$, is equicontinuous for any bounded set $A \subset \mathcal{D}'_{\Gamma_1}$. Because of the symmetry of the problem, we can prove similarly that the family of maps $u \times v \mapsto u \otimes v$, where $v \in B$, is equicontinuous for any bounded set $B \subset \mathcal{D}'_{\Gamma_2}$. Finally, we have proved

THEOREM 5.5. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$, $\Omega_2 \subset \mathbb{R}^{d_2}$ be open sets, $\Gamma_1 \in \dot{T}^*\Omega_1$, $\Gamma_2 \in \dot{T}^*\Omega_2$ be closed cones and*

$$\Gamma = (\Gamma_1 \times \Gamma_2) \cup ((\Omega_1 \times \{0\}) \times \Gamma_2) \cup (\Gamma_1 \times (\Omega_2 \times \{0\})).$$

Then, the tensor product $(u, v) \mapsto u \otimes v$ is hypocontinuous from $\mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2}$ to \mathcal{D}'_{Γ} , in the normal topology.

6. The pull-back

The purpose of this section is to prove

PROPOSITION 6.1. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and Γ a closed cone in $\dot{T}^*\Omega_2$. Let $f : \Omega_1 \rightarrow \Omega_2$ be a smooth map such that $N_f \cap \Gamma = \emptyset$, where $N_f = \{(f(x); \eta) \in \Omega_2 \times \mathbb{R}^n; \eta \circ df_x = 0\}$ and $f^*\Gamma = \{(x; \eta \circ df_x) | (f(x); \eta) \in \Gamma\}$. Then, the pull-back operation $f^* : \mathcal{D}'_{\Gamma}(\Omega_2) \rightarrow \mathcal{D}'_{f^*\Gamma}(\Omega_1)$ is continuous for the normal topology.*

We could prove that the pull-back is continuous by proving that $p_B(f^*u)$ is continuous for all the seminorms of the strong topology of \mathcal{D}' and then that $\|f^*u\|_{N, V, \chi}$ is continuous for every N and every V and χ such that $(\text{supp } \chi \times V) \cap f^*\Gamma = \emptyset$. However, it will be much simpler to use more topological methods and to just prove that $\langle f^*u, v \rangle$ is continuous for every v in an equicontinuous set. This is the purpose of the following section.

6.1. Equicontinuous bornology

Let Ω be open in \mathbb{R}^d and Γ be a closed cone in $\dot{T}^*\Omega$. We define the open cone $\Lambda = \{(x, \xi) \in \dot{T}^*\Omega; (x, -\xi) \notin \Gamma\}$ and the space $\mathcal{E}'_{\Lambda}(\Omega)$ of compactly supported distributions $v \in \mathcal{E}'(\Omega)$ such that $\text{WF}(v) \subset \Lambda$.

The following theorem will be useful to prove the continuity of linear maps [25, p. 200]:

THEOREM 6.2. — *If E is a locally convex space and $f : E \rightarrow \mathcal{D}'_\Gamma(\Omega)$ is a linear map, then f is continuous iff, for every equicontinuous set H of $\mathcal{E}'_\Lambda(\Omega)$ the seminorm $p_H : E \rightarrow \mathbb{R}$ defined by*

$$p_H(x) = \sup_{v \in H} |\langle f(x), v \rangle|,$$

is continuous.

Similar results exist for multilinear mappings [8]. Of course, this theorem can only be useful if the equicontinuous sets are known. Let us explain the concept of equicontinuity [25, p. 200] in the general context of a locally convex topological vector space E with seminorms $(p_\alpha)_{\alpha \in A}$. Let E^* be its topological dual, a set H in E^* is called *equicontinuous* iff the family of maps $\ell_v := u \mapsto \langle u, v \rangle$ is **uniformly** continuous when v runs over the set H . In other words, for every equicontinuous set $H \subset E^*$, there is a constant M and a finite family of seminorms $(p_\alpha)_{\alpha \in A}$ of E such that

$$(6.1) \quad \forall v \in H, \quad |\langle u, v \rangle| \leq M \left(\sum_{\alpha \in A} p_\alpha(u) \right).$$

Therefore a set H is equicontinuous in $\mathcal{E}'_\Lambda(\Omega)$ (which is the topological dual of $\mathcal{D}'_\Gamma(\Omega)$) iff there is a finite number of seminorms $\|\cdot\|_{N_1, V_1, \chi_1}, \dots, \|\cdot\|_{N_k, V_k, \chi_k}$ of $\mathcal{D}'_\Gamma(\Omega)$, a bounded subset B_0 of $\mathcal{D}(\Omega)$ and a constant M such that

$$(6.2) \quad |\langle u, v \rangle| \leq M \sup\{\|u\|_{N_1, V_1, \chi_1}, \dots, \|u\|_{N_k, V_k, \chi_k}, p_{B_0}(u)\}$$

for every $u \in \mathcal{D}'_\Gamma(\Omega)$ and every $v \in H$. There is only one seminorm p_{B_0} because these seminorms are saturated [25, p. 107] in $\mathcal{D}'(\Omega)$ with the strong topology.

We recall some useful terminology. Let X be a locally convex space and X' its topological dual. Then the strong topology $\beta(X', X)$ on X' is defined by the seminorms $P_B(y) = \sup_{x \in B} |\langle x, y \rangle|$ where B runs over all bounded sets in X . We also denote by π the projection: $(x; \xi) \in T^*\Omega \mapsto x \in \Omega$. The following lemma characterizes the equicontinuous sets of $\mathcal{E}'_\Lambda(\Omega)$:

LEMMA 6.3. — *A subset B of $\mathcal{E}'_\Lambda(\Omega)$ is equicontinuous iff there is: (i) a compact set $K \subset \Omega$ containing the support of all elements of B ; (ii) a closed cone $\Xi \subset \Lambda$ such that $B \subset \mathcal{D}'_\Xi(\Omega)$, B is bounded in $\mathcal{D}'_\Xi(\Omega)$ and $\pi(\Xi) \subset K$.*

Proof. — We first prove that every such B is equicontinuous. We showed in [9] that the space $\mathcal{E}'_\Lambda(\Omega)$ is the inductive limit of spaces $E_\ell = \{v \in \mathcal{E}'_\Lambda(\Omega); \text{supp } v \in L_\ell, \text{WF}(v) \in \Lambda_\ell\}$, where the compact sets L_ℓ exhaust Ω and the closed cones Λ_ℓ exhaust Λ . Thus, there is an integer ℓ such that $\Xi \subset \Lambda_\ell$ and $B \subset E_\ell$. The inclusion of Ξ in Λ_ℓ implies that every seminorm $\|\cdot\|_{N,V,\chi}$ of E_ℓ is also a seminorm of $\mathcal{D}'_\Xi(\Omega)$ because $\text{supp } \chi \times V$ does not meet Ξ if it does not meet Λ_ℓ . Thus, B is bounded in E_ℓ and Eq. (8) of [9] gives us

$$\sup_{v \in B} |\langle u, v \rangle| \leq \sum_j \left(p_{B_j}(u) + \|u\|_{m+n+1, V_j, \chi_j} C I_n^{n+1} + \|u\|_{n, V_j, \chi_j} M_{n, W_j, \chi_j} I_n^{2n} \right),$$

which can be converted to the equicontinuity condition (6.2).

To show the converse, we denote by B the set of all $v \in \mathcal{E}'_\Lambda(\Omega)$ that satisfy Eq. (6.2). Then, by following exactly the proof of Prop. 7 of [9], we obtain that the support of all elements of B is included in a compact set $K = \cup_j \text{supp } \chi_j \cup K'$, where K' is a compact set containing the support of all $f \in B_0$. Moreover, the wave front set of all elements of B is contained in $\Xi = \cup_j \text{supp } \chi_j \times (-V_j)$. It remains to show that B is bounded in $\mathcal{D}'_\Xi(\Omega)$ for the normal topology. We first notice that, if $\text{supp } \chi \times (-V) \subset \Xi$, then $\|\cdot\|_{N,V,\chi}$ is a continuous seminorm of the strong dual $\mathcal{E}'_{(\Xi')^c}(\Omega)$ of $\mathcal{D}'_\Xi(\Omega)$. Indeed, it was shown in the proof of Prop. 7 of [9] that $\|u\|_{N,V,\chi} = \sup_{\xi \in V} |\langle u, f_\xi \rangle|$, where $f_\xi(x) = (1 + |\xi|)^N \chi(x) e^{-i\xi \cdot x}$ and the set $\{f_\xi, \xi \in V\}$ is bounded in $\mathcal{D}'_\Xi(\Omega)$. If B' is a bounded set in $\mathcal{E}'_{(\Xi')^c}(\Omega)$, the continuous seminorms $\|u\|_{N,V,\chi}$ and $p_{B_0}(u)$ of $\mathcal{E}'_{(\Xi')^c}(\Omega)$ appearing on the right hand side of ((6.2)) are bounded over B' . Thus, for any bounded set B' in $\mathcal{E}'_{(\Xi')^c}(\Omega)$, taking $u \in B'$ and taking the sup in ((6.2)) over $u \in B'$ yields that $\sup_{u \in B', v \in B} |\langle u, v \rangle|$ is bounded and B is a bounded subset of $\mathcal{D}'_\Xi(\Omega)$ when $\mathcal{D}'_\Xi(\Omega)$ is equipped with the strong $\beta(\mathcal{D}'_\Xi, \mathcal{E}'_{(\Xi')^c})$ topology. It is shown in [9, Theorem 33] that the bounded sets of $\mathcal{D}'_\Gamma(\Omega)$ coincide for the strong and the normal topologies. Thus, B is bounded for the normal topology. \square

We obtain the following characterization of continuous linear maps:

THEOREM 6.4. — *Let E be a locally convex space, Ω an open subset of \mathbb{R}^d and Γ a closed cone in $\dot{T}^*\Omega$. A linear map $f : E \rightarrow \mathcal{D}'_\Gamma(\Omega)$ is continuous iff every map $f_B : E \rightarrow \mathbb{R}$ defined by $f_B(x) = \sup_{v \in B} |\langle f(x), v \rangle|$ is continuous, where B is equicontinuous in $\mathcal{E}'_\Lambda(\Omega)$, $\Lambda = (\Gamma')^c$.*

Recall $\Lambda = (\Gamma')^c$. The equicontinuous sets of $\mathcal{E}'_\Lambda(\Omega)$ intervene also because of the following theorem [26, p. 157]:

THEOREM 6.5. — *Let the duality pairing $\mathcal{D}'_\Gamma(\Omega) \times \mathcal{E}'_\Lambda(\Omega) \rightarrow \mathbb{K}$ be defined by $u \times v \rightarrow f(u, v) = \langle u, v \rangle$. Then, for every bounded set A of $\mathcal{D}'_\Gamma(\Omega)$ and*

every equicontinuous set B of $\mathcal{E}'_{\Lambda}(\Omega)$ the sets of maps $\{f_u; u \in A\}$ and $\{f_v; v \in B\}$ are equicontinuous.

We cannot expect the duality pairing to be (jointly) continuous because this happens only for normable spaces [25, p. 359]. The spaces $\mathcal{D}'_{\Gamma}(\Omega)$ and $\mathcal{E}'_{\Lambda}(\Omega)$ are not normable. In fact, a nuclear space is normable if and only if it is finite dimensional [36, p. 520].

Strategy of proof that the pull-back is continuous. Let Ω_1 and Ω_2 be open sets in \mathbb{R}^{d_1} and \mathbb{R}^{d_2} , respectively. Let $f : \Omega_1 \rightarrow \Omega_2$ be a smooth map and Γ be a closed cone in $\dot{T}^*\Omega_2$. We want to show that the pull-back $f^* : \mathcal{D}'_{\Gamma}(\Omega_2) \rightarrow \mathcal{D}'_{f^*\Gamma}(\Omega_1)$ is continuous for the normal topology. According to Theorem 6.4, the pull-back is continuous iff, for every equicontinuous set $B \subset \mathcal{E}'_{\Lambda}(\Omega_1)$ (where $\Lambda = (f^*\Gamma)'^c$) the family of maps $(\rho_v)_{v \in B}$, defined by $\rho_v : u \mapsto \langle f^*u, v \rangle$, is equicontinuous which implies that $\sup_{v \in B} |\langle f^*u, v \rangle|$ is continuous in u . By Lemma 6.3, we know that there is a compact set $K \subset \Omega_1$ and a closed cone $\Xi \subset (f^*\Gamma')^c$ such that $\text{supp } v \subset K$ and $\text{WF}(v) \subset \Xi$ for all $v \in B$. Choose a function $\chi \in \mathcal{D}(\Omega_1)$ such that $\chi|_K = 1$. If $(\varphi_i)_{i \in I}$ is a partition of unity of Ω_2 , we can write $\langle f^*u, v \rangle = \sum_i \langle f^*(u\varphi_i), v\chi \rangle$. The image of $\text{supp } \chi$ by f being compact [4, p. 19], only a finite number of terms of this sum are nonzero and the family ρ_v is equicontinuous iff, for every $\varphi \in \mathcal{D}(\Omega_2)$, the family of maps $u \mapsto \langle f^*(u\varphi), v\chi \rangle$ is equicontinuous.

Stationary phase and Schwartz kernels. In order to calculate the pairing between $f^*(u\varphi)$ and v , we first notice that, when u is a locally integrable function, then $u\varphi(y) = \mathcal{F}^{-1}(\widehat{u\varphi})(y) = (2\pi)^{-d_2} \int_{\mathbb{R}^{d_2}} d\eta e^{i\eta \cdot y} \widehat{u\varphi}(\eta)$, so that

$$f^*(u\varphi)(x) = (2\pi)^{-d_2} \int_{\mathbb{R}^{d_2}} d\eta e^{i\eta \cdot f(x)} \widehat{u\varphi}(\eta)$$

and

$$\begin{aligned} \langle f^*(u\varphi), \chi v \rangle &= \frac{1}{(2\pi)^{d_2}} \int_{\Omega_1} \int_{\mathbb{R}^{d_2}} \chi(x) v(x) e^{i\eta \cdot f(x)} \widehat{u\varphi}(\eta) dx d\eta \\ &= \frac{1}{(2\pi)^{d_2}} \int_{\mathbb{R}^{d_2}} \int_{\Omega_1} \int_{\mathbb{R}^{d_2}} \chi(x) v(x) e^{i\eta \cdot f(x)} e^{-iy \cdot \eta} u(y) \varphi(y) dy dx d\eta. \end{aligned}$$

This definition can be extended to any distribution $u \in \mathcal{D}'_{\Gamma}$ as

$$(6.3) \quad \langle f^*(u\varphi), \chi v \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} u(y) v(x) I(x, y) dx dy,$$

where $d = d_1 + d_2$, $I(x, y) = \int_{\mathbb{R}^{d_2}} e^{i\eta \cdot (f(x) - y)} \varphi(y) \chi(x) d\eta$. The duality pairing can also be written

$$(6.4) \quad \langle f^*(u\varphi), v\chi \rangle = \langle v \otimes u, I \rangle.$$

Note that $I(x, y) = (2\pi)^{-d_2} \chi(x) \varphi(y) \int d\eta e^{i\eta \cdot (f(x) - y)}$ is an oscillatory integral in the sense of Hörmander [24][32] with symbol $\chi(x)\varphi(y)$ and phase $\eta \cdot (f(x) - y)$ where $\eta \cdot (f(x) - y)$ is homogeneous of degree 1 w.r.t. η , $\forall \eta \neq 0, d(\eta \cdot (f(x) - y)) \neq 0$. Therefore, $I \in \mathcal{D}'(\Omega_1 \times \Omega_2)$ is the **Schwartz kernel of the bilinear continuous map**:

$$(6.5) \quad (u, v) \in \mathcal{D}(\Omega_1) \times \mathcal{D}(\Omega_2) \mapsto \langle f^*(u\varphi), v\chi \rangle.$$

Proof of Proposition 6.1. By Theorem 5.5, the map $(v, u) \mapsto v \otimes u$ is hypocontinuous from $\mathcal{D}'_{\Xi} \times \mathcal{D}'_{\Gamma}$ to $\mathcal{D}'_{\Gamma \otimes}$ where $\Gamma \otimes = \Xi \times \Gamma \cup (\Omega_1 \times \{0\}) \times \Gamma \cup \Xi \times (\Omega_2 \times \{0\})$. Let Λ_{\otimes} be the open cone $\Gamma'_{\otimes, c}$. Therefore by Theorem 6.5, the family of duality pairings

$$u \otimes v \in \mathcal{D}'_{\Gamma \otimes} \mapsto \langle u \otimes v, r \rangle$$

is equicontinuous from $\mathcal{D}'_{\Gamma \otimes}$ to \mathbb{K} uniformly in $r \in B'$ for every equicontinuous set B' of $\mathcal{E}'_{\Lambda_{\otimes}}$. In particular, if B' contains only the element I , then the map $v \otimes u \mapsto \langle v \otimes u, I \rangle$ would be continuous if I were compactly supported, which is the case because its support is included in $\text{supp } \chi \times \text{supp } \varphi$ and if its wave front set were contained in Λ_{\otimes} . Thus, if $\text{WF}(I) \subset \Lambda_{\otimes}$, then the map $(v, u) \mapsto \langle v \otimes u, I \rangle$ is hypocontinuous because, by the next lemma, the composition of a hypocontinuous map by a continuous map is hypocontinuous. In other words, the map $(v, u) \mapsto \langle f^*(u\varphi), \chi v \rangle$ is hypocontinuous, by item (i) of Definition 4.2, this implies that the family of maps $\rho_v : u \mapsto \langle f^*(u\varphi), \chi v \rangle$ with $v \in B$ is equicontinuous.

LEMMA 6.6. — *The composition of a hypocontinuous map by a continuous map is hypocontinuous.*

This result is known [18] but we could not find a proof in the literature.

Proof. — Let $f : E \times F \rightarrow G$ be a hypocontinuous map and $g : G \rightarrow H$ a continuous map. The map $g \circ f$ is hypocontinuous iff, for every bounded set $B \subset F$ and every neighborhood W of zero in H , there is a neighborhood U of zero in E such that $(g \circ f)(U \times B) \subset W$ (with the similar condition for $(g \circ f)(A \times V)$) [3, p. III.30]. By the continuity of g , there is a neighborhood Z of zero in G such that $g(Z) \subset W$. By the hypocontinuity of f , there is a neighborhood U of zero in E such that $f(U \times B) \subset Z$. Thus, $(g \circ f)(U \times B) \subset g(Z) \subset W$. \square

It just remains to check that $\text{WF}(I) \subset \Lambda_\otimes$, i.e. that $\text{WF}(I)'$ does not meet Γ_\otimes . The wave front set of I is $\text{WF}(I) \subset \{(x, f(x); -\eta \circ d_x f, \eta); x \in \text{supp } \chi\}$ [24, p. 260]. Recall that $\Xi \subset (f^* \Gamma')^c = \{(x, -\eta \circ d_x f); (f(x), \eta) \notin \Gamma\}$. By definition of Γ_\otimes we must satisfy the following three conditions:

- $\Xi \times \Gamma \cap \text{WF}(I)' = \emptyset$ because it is the set of points $(x, f(x); -\eta \circ d_x f, \eta)$ such that $(f(x), \eta) \notin \Gamma$ by definition of Ξ and $(f(x), \eta) \in \Gamma$ by definition of Γ ;
- $\Xi \times (\Omega_2 \times \{0\}) \cap \text{WF}(I)' = \emptyset$ because we would need $\eta = 0$ whereas $(y, \eta) \in \Gamma$ implies $\eta \neq 0$;
- $(\text{supp } \chi \times \{0\}) \times \Gamma \cap \text{WF}(I)' \subset \{(x, f(x); 0, \eta); x \in \text{supp } \chi, \eta \circ d_x f = 0, (f(x), \eta) \in \Gamma\}$.

Thus, if $f^* \Gamma \cap N_f = \emptyset$, then $\text{WF}(I)' \cap \Gamma_\otimes = \emptyset$ and the pull-back is continuous.

How to write the pull-back operator in terms of the Schwartz kernel I ? Relationship with the product of distributions. We start from a linear operator $L : \mathcal{D}(\mathbb{R}^{d_2}) \mapsto \mathcal{D}'(\mathbb{R}^{d_1})$ with corresponding Schwartz kernel $K \in \mathcal{D}'(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Using the standard operations on distributions, we can make sense of the well-known representation formula $Lu = \int_{\mathbb{R}^{d_2}} K(x, y) u(y) dy$ for an operator $L : \mathcal{D}(\mathbb{R}^{d_2}) \mapsto \mathcal{D}'(\mathbb{R}^{d_1})$ and its kernel $K \in \mathcal{D}'(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Let us define the two projections $\pi_2 := (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \mapsto y \in \mathbb{R}^{d_2}$ and $\pi_1 := (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \mapsto x \in \mathbb{R}^{d_1}$, then we define $K(x, y)u(y) = K(x, y)\pi_2^* u(x, y) = K(x, y)(1(x) \otimes u(y))$ where $\pi_2^* u = 1 \otimes u$ and $\int_{\mathbb{R}^{d_2}} dy f(x, y) = \pi_{1*} f(x)$. Therefore

$$(6.6) \quad Lu = \int_{\mathbb{R}^{d_2}} K(x, y) u(y) dy = \pi_{1*} (K (\pi_2^* u)).$$

The interest of the formula $Lu = \pi_{1*} (K (\pi_2^* u))$ is that everything generalizes to oriented manifolds. Replace \mathbb{R}^{d_2} (resp \mathbb{R}^{d_1}) with a manifold M_2 (resp M_1) with smooth volume densities $|\omega_2|$ (resp $|\omega_1|$), the duality pairing is defined as the extension of the usual integration against the volume densities, for instance: $\forall (u, \varphi) \in C^\infty(M_1) \times \mathcal{D}(M_1)$, $\langle u, \varphi \rangle_{M_1} = \int_{M_1} (u\varphi)\omega_1$. Finally, for the linear continuous map $L : u \in \mathcal{D}(\mathbb{R}^{d_2}) \mapsto \chi f^*(u\varphi) \in \mathcal{D}'(\mathbb{R}^{d_1})$, we get the formula:

$$(6.7) \quad Lu = \pi_{1*} (I(\pi_2^* u))$$

where $I(x, y) = (2\pi)^{-d_2} \chi(x) \varphi(y) \int d\eta e^{i\eta \cdot (f(x) - y)}$ is the Schwartz kernel of L .

6.2. Pull-back by families of smooth maps

LEMMA 6.7. — *Let Ω_1, Ω_2, U be open sets in $\mathbb{R}^{d_1}, \mathbb{R}^{d_2}, \mathbb{R}^n$ respectively. For any compact sets $(K_1 \subset \Omega_1, K_2 \subset \Omega_2, A \subset U)$, the conic set*

$\Gamma = \{(x, f(x, a); -\eta \circ d_x f(x, a), \eta); (x, a, f(x, a)) \in K_1 \times A \times K_2, \eta \neq 0\}$
is closed in $\dot{T}^(\Omega_1 \times \Omega_2)$.*

Proof. — Let $(x, y; \xi, \eta) \in \bar{\Gamma}$ such that $(\xi, \eta) \neq (0, 0)$. Then there is a sequence

$$(x_n, f(x_n, a_n); -\eta_n \circ d_x f(x_n, a_n), \eta_n) \in \Gamma, (x_n, a_n, f(x_n, a_n)) \in K_1 \times A \times K_2$$

which converges to $(x, y; \xi, \eta)$. By compactness of A , we extract a convergent subsequence $a_n \rightarrow a$. By continuity of $d_x f$, we find that $\xi = -\eta \circ d_x f(x, a)$, we also find that $\lim_{n \rightarrow \infty} f(x_n, a_n) = f(x, a) \in K_2$ since K_2 is closed and we finally note that we must have $\eta \neq 0$ otherwise $\xi = 0, \eta = 0$. Therefore $(x, y; \xi, \eta) \in \Gamma$ by definition. Finally, $\bar{\Gamma} \subset \Gamma$ hence Γ is closed. \square

PROPOSITION 6.8. — *Let Ω_1 be an open set in \mathbb{R}^{d_1} , $A \subset U \subset \mathbb{R}^n$ where A is compact, U and Ω_2 are open sets in \mathbb{R}^{d_2} . Let $\chi \in \mathcal{D}(\Omega_1)$, $\varphi \in \mathcal{D}(\Omega_2)$ and f a smooth map $f : \Omega_1 \times U \rightarrow \Omega_2$.*

(1) *Then the family of distributions $(I_{f(.,a)})_{a \in A}$ formally defined by*

$$I_{f(.,a)}(x, y) = \chi(x)\varphi(y) \int_{\mathbb{R}^{d_2}} \frac{d\theta}{(2\pi)^{d_2}} e^{i\theta \cdot (f(x,a) - y)}$$

is a bounded set in \mathcal{D}'_Γ , where Γ is the closed cone in $\dot{T}^(\Omega_1 \times \Omega_2)$ defined by:*

$$\begin{aligned} \Gamma = \{ & (x, f(x, a); -\eta \circ d_x f(x, a), \eta); \\ & x \in \text{supp } \chi, f(x, a) \in \text{supp } \varphi, a \in A, \eta \neq 0\}. \end{aligned}$$

(2) *For any open cone Λ containing Γ , $(I_{f(.,a)})_{a \in A}$ is equicontinuous in $\mathcal{E}'_\Lambda(\Omega_1 \times \Omega_2)$.*

We will use the pushforward theorem 7.3 in the following proof.

Proof. — From Lemma 6.3 and from the fact that $(I_{f(.,a)})_{a \in A}$ is supported in a fixed compact set $\text{supp } \chi \times \text{supp } \varphi$, we deduce that conclusion (2) follows from the first claim thus it suffices to prove the claim (1).

The conic set Γ is closed by Lemma 6.7. To prove that the family $(I_{f(.,a)})_{a \in A}$ is bounded in \mathcal{D}'_Γ , it suffices to check that

$$\forall v \in \mathcal{E}'_{\Gamma^c}(\Omega_1 \times \Omega_2), \sup_{a \in A} |\langle I_{f(.,a)}, v \rangle| < +\infty$$

because of [9, Proposition 1].

Step 1 Our goal is to study the map $a \mapsto \int_{\Omega_1 \times \Omega_2} I_f(x, y, a) v(x, y)$ where $I_f(x, y, a) = \chi(x) \varphi(y) \int_{\mathbb{R}^{d_2}} \frac{d\theta}{(2\pi)^{d_2}} e^{i\theta \cdot (f(x, a) - y)}$. Let π_{12}, π_3 be projections defined by the formulas

$$\begin{aligned} \pi_{12} : (x, y, a) \in \Omega_1 \times \Omega_2 \times U &\longmapsto (x, y) \in \Omega_1 \times \Omega_2 \\ \text{and } \pi_3 : (x, y, a) \in \Omega_1 \times \Omega_2 \times U &\longmapsto a \in U. \end{aligned}$$

Using the dictionary explained in paragraph 6.1, if v were a test function, then we would find that

$$(6.8) \quad \int_{\Omega_1 \times \Omega_2} I_f(x, y, \cdot) v(x, y) dx dy = \pi_{3*} (I_f \pi_{12}^* v) \in \mathcal{D}'(U).$$

We want to prove that $a \mapsto \int_{\Omega_1 \times \Omega_2} I_f(x, y, a) v(x, y) dx dy$ is smooth in some open neighborhood of A since this would imply that

$$\sup_{a \in A} \left| \int_{\Omega_1 \times \Omega_2} I_f(x, y, a) v(x, y) dx dy \right| = \sup_{a \in A} |\langle I_f(\cdot, a), v \rangle| < +\infty.$$

In order to do so, it suffices to prove that the condition $v \in \mathcal{E}'_{\Gamma', c}$ implies that the distributional product $I_f(x, y, a) v(x, y) = I_f(\pi_{12}^* v)(x, y, a)$ makes sense in $\mathcal{D}'(\Omega_1 \times \Omega_2 \times U)$ and the push-forward $\pi_{3*} (I_f \pi_{12}^* v) = \int_{\Omega_1 \times \Omega_2} I_f(x, y, \cdot) v(x, y) dx dy$ has empty wave front set over some open neighborhood of A .

Step 2 The wave front sets of I_f and $\pi_{12}^* v$ are:

$$\begin{aligned} WF(I_f) &= \left\{ \begin{pmatrix} x & ; & -\theta \circ d_x f \\ f(x, a) & ; & \theta \\ a & ; & -\theta \circ d_a f \end{pmatrix} ; \right. \\ &\quad \left. x \in \text{supp } \varphi, f(x, a) \in \text{supp } \chi, a \in U, \theta \neq 0 \right\} \\ WF(\pi_{12}^* v) = WF(v \otimes 1) &= \left\{ \begin{pmatrix} x & ; & \xi \\ y & ; & \eta \\ a & ; & 0 \end{pmatrix} ; \begin{pmatrix} x & ; & \xi \\ y & ; & \eta \end{pmatrix} \in WF(v) \right\}. \end{aligned}$$

One also have

$$\begin{aligned} v \in \mathcal{E}'_{\Gamma', c} &\implies WF(v) \cap \Gamma' = \emptyset \\ &\implies \xi \neq -\eta \circ d_x f \\ &\implies \forall \theta, \begin{pmatrix} \xi - \theta \circ d_x f \\ \theta + \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ has no solution} \end{aligned}$$

Observe that

$$WF(I_f) + WF(\pi_{12}^* v) = \left\{ \begin{pmatrix} x & ; & \xi - \theta \circ d_x f \\ f(x) & ; & \theta + \eta \\ a & ; & -\theta \circ d_a f \end{pmatrix}, \forall \theta \in \mathbb{R}^d \setminus \{0\} \right\}$$

$$\begin{aligned} \implies & (WF(I_f) + WF(\pi_{12}^* v)) \cap \underline{0} = \emptyset \\ \text{and} & (WF(I_f) \cup WF(\pi_{12}^* v)) \cap \underline{0} = \emptyset. \end{aligned}$$

Step 3 In the last step, we shall prove that the condition $WF(v) \cap \Gamma' = \emptyset$ actually implies that $WF(\pi_{3*}(I_f \pi_{12}^* v))$ is empty over some open neighborhood U' of A . The condition $WF(v) \cap \Gamma' = \emptyset$ implies the existence of some open neighborhood U' of A s.t.

$$\forall a \in U', \forall (x, f(x, a); \xi, \eta) \in WF(v), \xi \neq -\eta \circ d_x f(x, a).$$

Since A and $\text{supp } v$ are compact and $A \times WF(v)$ is closed, we can find $\delta > 0$ s.t.

$$\forall (a, (x, f(x, a); \xi, \eta)) \in A \times WF(v), |\xi + \eta \circ d_x f(x, a)| \geq \delta |\eta|.$$

Define $U' = \{a \in U; \forall (x, f(x, a); \xi, \eta) \in WF(v), |\xi + \eta \circ d_x f(x, a)| > \frac{\delta}{2} |\eta|\}$. Therefore, the condition $WF(v) \cap \Gamma' = \emptyset$ on the wave front set of v ensures that $\pi_{3*}(I_f \pi_{12}^* v)$ is well defined in $\mathcal{D}'_\emptyset(U') = C^\infty(U')$. Hence $a \mapsto \langle v, I_{f(\cdot, a)} \rangle = \int_{\Omega_1 \times \Omega_2} I_f(x, y, a) v(x, y) dx dy$ is smooth on U' , a fortiori continuous on the compact set A which means that

$$\sup_{a \in A} |\langle v, I_{f(\cdot, a)} \rangle| < +\infty.$$

□

THEOREM 6.9. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, $A \subset U \subset \mathbb{R}^n$ where A compact, U open and Γ a closed cone in $\dot{T}^*\Omega_2$. Let $f : \Omega_1 \times U \rightarrow \Omega_2$ be a smooth map such that $\forall a \in A, f(\cdot, a)^* \Gamma$ does not meet the zero section $\underline{0}$ and set $\Theta = \bigcup_{a \in A} f(\cdot, a)^* \Gamma$. Then for all seminorms P_B of $\mathcal{D}'_\Theta(\Omega_1)$, $\forall u \in \mathcal{D}'_\Gamma(\Omega_2)$, the family $P_B(f(\cdot, a)^* u)_{a \in A}$ is bounded.*

Proof. — Let B be equicontinuous in $\mathcal{E}'_{\Theta, c}(\Omega_1)$. We need to prove that $\sup_{(v, a) \in B \times A} |\langle f(\cdot, a)^* u, v \rangle| < +\infty$. It follows from Lemma 6.3 that there exists some closed cone Ξ such that $\Xi \cap \Theta' = \emptyset$ and $B \subset \mathcal{D}'_\Xi(\Omega_1)$. Set $\Gamma_\otimes = (\Xi \times \Gamma) \cup ((\Omega_1 \times \{0\}) \times \Gamma) \cup (\Xi \times (\Omega_2 \times \{0\}))$. Let Λ_\otimes be the open cone defined as $\Lambda_\otimes = (\Gamma_\otimes)'^{c, c}$. By proposition 6.8, we can easily verify as in the proof of Proposition 6.1 that the family $(I_{f(\cdot, a)})_{a \in A}$ is equicontinuous in $\mathcal{E}'_{\Lambda_\otimes}$. Then we prove similarly as in the proof of the pull-back Proposition 6.1 that the family of maps $\rho_{v, a} : u \mapsto \langle f(\cdot, a)^* (u\varphi), \chi v \rangle$ with $(v, a) \in B \times A$ is equicontinuous where B is equicontinuous in $\mathcal{E}'_{\Theta, c}(\Omega_1)$ and χ is chosen in such a way that $\chi|_{\text{supp } B} = 1$ and $\varphi|_{f(\text{supp } B)} = 1$, therefore:

$$\sup_{(v, a) \in B \times A} |\langle f(\cdot, a)^* u, v \rangle| = \sup_{(v, a) \in B \times A} |\langle f(\cdot, a)^* (u\varphi), \chi v \rangle| < +\infty.$$

and the result follows from the characterization of continuous linear maps by Theorem 6.4. □

7. Product, convolution and push-forward.

Hörmander noticed that the product of distributions u and v can be described as the composition of the tensor product $(u, v) \rightarrow u \otimes v$ with the pull-back by the map $f : x \mapsto (x, x)$. If the wave front sets of u and v are contained in Γ_1 and Γ_2 , then the wave front set of $u \otimes v$ is contained in $\Gamma_\otimes = (\Gamma_1 \times \Gamma_2) \cup ((\Omega_1 \times \{0\}) \times \Gamma_2) \cup (\Gamma_1 \times (\Omega_2 \times \{0\}))$ and the pull-back is well-defined if the set $N_f = \{(x, x; \eta_1, \eta_2); (\eta_1 + \eta_2) \circ dx = 0\}$, which is the conormal bundle of the diagonal $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$ as in example 2.9, does not meet Γ , i.e. if there is no point $(x; \eta)$ in Γ_1 such that $(x; -\eta)$ is in Γ_2 . Therefore, the multiplication of distributions is hypocontinuous because it is the composition of a hypocontinuous map by a continuous map (see Lemma 6.6).

THEOREM 7.1. — *Let $\Omega \subset \mathbb{R}^n$ be an open set and Γ_1, Γ_2 be two closed cones in $\dot{T}^*\Omega$ such that $\Gamma_1 \cap \Gamma_2' = \emptyset$. Then the product of distributions is hypocontinuous for the normal topology from $\mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2}$ to \mathcal{D}'_Γ , where*

$$(7.1) \quad \Gamma = (\Gamma_1 +_\Omega \Gamma_2) \cup ((\Omega \times \{0\}) +_\Omega \Gamma_2) \cup (\Gamma_1 +_\Omega (\Omega \times \{0\})).$$

Proof. — The product of distribution is the composition of the hypocontinuous tensor product with the continuous pull-back. \square

This gives us the useful corollary:

LEMMA 7.2. — *Let $\Omega \subset \mathbb{R}^n$ be an open set and Γ be a closed cone in $\dot{T}^*\Omega$. Then the product of a smooth map and a distribution is hypocontinuous for the normal topology from $C^\infty(\Omega) \times \mathcal{D}'_\Gamma$ to \mathcal{D}'_Γ .*

Proof. — We prove in section 10.3 that $C^\infty(\Omega)$ and \mathcal{D}'_\emptyset are topologically isomorphic. Therefore, the corollary follows by applying Theorem 7.1 to $\Gamma_1 = \emptyset$ and $\Gamma_2 = \Gamma$. Equation (7.1) shows that the wave front set of the product is in Γ . \square

7.1. The push-forward as a consequence of the pull-back theorem.

THEOREM 7.3. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and Γ a closed cone in $\dot{T}^*\Omega_1$. For any smooth map $f : \Omega_1 \rightarrow \Omega_2$ and any closed subset C of Ω_1 such that $f|_C : C \rightarrow \Omega_2$ is proper and $\underline{\pi}(\Gamma) \subset C$, then f_* is continuous in the normal topology from $\{u \in \mathcal{D}'_\Gamma; \text{supp } u \subset C\}$ to $\mathcal{D}'_{f_*\Gamma}$, where $f_*\Gamma = \{(y; \eta) \in \dot{T}^*\Omega_2; \exists x \in \Omega_1 \text{ with } y = f(x) \text{ and } (x; \eta \circ df_x) \in \Gamma \cup \text{supp } u \times \{0\}\}$.*

Proof. — The idea of the proof is to think of a push-forward as the adjoint of a pull-back (see [13]). For all B equicontinuous in \mathcal{E}'_Λ where $\Lambda = f_*\Gamma'^c$, $\forall v \in B$, $\text{supp } (f^*v) \cap C$ is contained in a fixed compact set K since f is proper on C . Hence, for any $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\chi|_C = 1$ and f is proper on $\text{supp } \chi$, we should have at least formally $\langle f_*u, v \rangle = \langle u, \chi f^*v \rangle$ if the duality couplings make sense. On the one hand, if $v \in \mathcal{E}'_\Lambda(\Omega_2)$ where $f^*\Lambda$ does not meet the zero section $\underline{0} \subset T^*\Omega_1$ then the pull-back f^*v would be well defined by the pull-back Proposition (6.1) (which is equivalent to the fact that $N_f \cap \Lambda = \emptyset$). On the other hand, the duality pairing $\langle u, \chi(f^*v) \rangle$ is well defined if $f^*\Lambda \cap \Gamma' = \emptyset$. Combining both conditions leads to the requirement that $f^*\Lambda \cap (\Gamma' \cup \underline{0}) = \emptyset$. But note that:

$$\begin{aligned} & (f^*\Lambda) \cap (\Gamma' \cup \underline{0}) = \emptyset \\ \Leftrightarrow & \{(x; \eta \circ df) \mid (f(x); \eta) \in \Lambda, (x; \eta \circ df) \in \Gamma' \cup \{0\}\} = \emptyset \\ \Leftrightarrow & (f(x); \eta) \in \Lambda \implies (x; \eta \circ df) \notin \Gamma' \cup \{0\} \\ \Leftrightarrow & (f(x); \eta) \in \Lambda' \implies (x; \eta \circ df) \notin \Gamma \cup \{0\}. \end{aligned}$$

which is equivalent to the fact that Λ' does not meet $f_*\Gamma = \{(f(x); \eta); (x; \eta \circ df) \in (\Gamma \cup \underline{0}), \eta \neq 0\} \subset T^*\Omega_2$ which is exactly the assumption of our theorem. Therefore, the set of distributions χf^*B is supported in a fixed compact set, bounded in $\mathcal{D}'_{f^*\Lambda}(\Omega_1)$ by the pull-back Proposition 6.1 applied to f^* , the formal duality couplings are well defined and:

$$\sup_{v \in B} |\langle f_*u, v \rangle| = \sup_{v \in B'} |\langle u, v \rangle|$$

where $B' = \chi f^*B$ is equicontinuous in $\mathcal{E}'_{\Gamma',c}(\Omega_1)$ (the support of the distribution is compact because f is proper) which means that $\sup_{v \in B'} |\langle u, v \rangle|$ is a continuous seminorm for the normal topology of $\mathcal{D}'_\Gamma(\Omega_1)$. \square

We state and prove a parameter version of the push-forward theorem

THEOREM 7.4. — *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, $A \subset U \subset \mathbb{R}^n$ where A is compact, U is open and Γ a closed cone in $T^*\Omega_1$. For any smooth map $f : \Omega_1 \times U \rightarrow \Omega_2$ and any closed subset C of Ω_1 such that $f : C \times A \rightarrow \Omega_2$ is proper and $\pi(\Gamma) \subset C$, then $f(\cdot, a)_*$ is uniformly continuous in the normal topology from $\{u \in \mathcal{D}'_\Gamma; \text{supp } u \subset C\}$ to \mathcal{D}'_Ξ , where $\Xi = \cup_{a \in A} f(\cdot, a)_*\Gamma$.*

Proof. — We have to check that Ξ is closed over $\dot{T}^*_{f(C \times A)}\Omega_2$. Let $(y; \eta) \in \Xi \cap \dot{T}^*_{f(C \times A)}\Omega_2$ then there exists a sequence $(y_n; \eta_n) \rightarrow (y; \eta)$ such that $(y_n; \eta_n) \in \Xi \cap \dot{T}^*_{f(C \times A)}\Omega_2$. By definition, $y_n = f(x_n, a_n)$ where $(x_n, \eta_n \circ d_x f(x_n, a_n)) \in \Gamma \cup \underline{0}$, $(x_n, a_n) \in C \times A$. The central observation is that

$\overline{\{y_n | n \in \mathbb{N}\}} \subset f(C \times A)$ and $\overline{\{(x_n, a_n) | f(x_n, a_n) = y_n, n \in \mathbb{N}\}} \subset C \times A$ are compact sets because f is proper on $C \times A$. Then we can extract convergent subsequences $(x_n, a_n) \rightarrow (x, a)$ and $(x, \eta \circ d_x f(x, a)) \in \Gamma \cup \underline{Q}$ since $\Gamma \cup \underline{Q}$ is closed in $\dot{T}^*\Omega_2$ and $d_x f$ is continuous. By definition of Ξ , this proves that $(y; \eta) \in \Xi$ and we can conclude that Ξ is closed. Now we can repeat the proof of the push-forward theorem except that we use the pull-back theorem with parameters. Let B be equicontinuous in $\mathcal{E}'_{\Xi, c}(\Omega_2)$ hence all elements of B have support contained in some compact. We have $\forall v \in B, \text{supp}(f^*v) \cap C \times A$ is compact, therefore for $\chi = 1$ on $\cup_{a \in A} \text{supp}(f(\cdot, a)^*v) \cap C$, the family $B' = \{(\chi f(\cdot, a)^*v) | a \in A, v \in B\}$ is equicontinuous in \mathcal{E}'_Θ where $\Theta = \Gamma'^c$ by the parameter version of the pull-back theorem. Therefore, $u \mapsto \sup_{a \in A} \sup_{v \in B} |\langle f(\cdot, a)_* u, v \rangle| = \sup_{v \in B'} |\langle u, v \rangle|$ is continuous in u since the right hand term is a continuous seminorm for the normal topology of $\mathcal{D}'_\Gamma(\Omega_1)$. \square

Convolution product. In the same spirit as for the multiplication of distributions, Duistermaat–Kolk [14, p. 121] describe the convolution product $u * v$ as the composition of the tensor product $(u, v) \mapsto u \otimes v$ with the push-forward by the map $\Sigma := (x, y) \mapsto (x + y)$. For a closed subset $X \subset \mathbb{R}^n$, let $\mathcal{D}'_\Gamma(X)$ be the set of distributions supported in X with wave front in Γ . Therefore, we have the

THEOREM 7.5. — *Let Γ_1, Γ_2 be two closed conic sets in $\dot{T}^*\mathbb{R}^n$ and X_1, X_2 two closed subsets of \mathbb{R}^n such that $\Sigma : X_1 \times X_2 \mapsto \mathbb{R}^n$ is proper. Then the convolution product of distributions is hypocontinuous from $\mathcal{D}'_{\Gamma_1}(X_1) \times \mathcal{D}'_{\Gamma_2}(X_2)$ to $\mathcal{D}'_\Gamma(X_1 + X_2)$ where*

$$(7.2) \quad \Gamma = \{(x + y; \eta); (x; \eta) \in \Gamma_1, (y; \eta) \in \Gamma_2\}.$$

Proof. — The convolution product of distribution is the composition of the hypocontinuous tensor product with the continuous push-forward. \square

8. Coordinate invariant definition of the wave front set.

Duistermaat [13, p. 13] proposed a coordinate invariant definition of the wave front set that corrects a first attempt by Gabor [17].

DEFINITION 8.1. — *Let $\Omega \subset \mathbb{R}^d$ be an open set, $u \in \mathcal{D}'(\Omega)$. An element $(x; \xi) \notin WF_D(u)$ iff for all $f \in C^\infty(\Omega \times \mathbb{R}^n)$ such that $d_x f(x, a_0) = \xi$ for some $a_0 \in \mathbb{R}^n$, there exists some neighborhoods A of a_0 and U of x_0 such that for all $\varphi \in \mathcal{D}(U)$: $|\langle u, \varphi e^{i\tau f(\cdot, a)} \rangle| = O(\tau^{-\infty})$ uniformly in some neighborhood of a_0 in A .*

The definition 8.1 looks a priori stronger than the property of being in the complementary of $WF(u)$. It is however equivalent. We give here an alternative proof.

THEOREM 8.2. — $WF_D(u) = WF(u)$.

Proof. — The inclusion $WF(u) \subset WF_D(u)$ is easy. If $(x; \xi) \notin WF_D(u)$, by applying Duistermaat's definition to the Fourier phase $f(x, a) = \langle x, a \rangle$, $a \in \mathbb{R}^d$, $a_0 = \xi$, A is a conic neighborhood of $a_0 = \xi$ and U is some neighborhood of x such that $(U \times A) \cap WF_D(u) = \emptyset$, we find that $(x; \xi) \notin WF(u)$.

The converse follows from the Proposition 8.3 below. \square

For any smooth map $f \in C(\mathbb{R}^d, \mathbb{R})$, we denote by $\mathbf{Gr}d_x f$ the graph of $d_x f$ in $T^*\mathbb{R}^d$.

PROPOSITION 8.3. — *Let Ω be an open set in \mathbb{R}^d , $A \subset U \subset \mathbb{R}^n$ where A is compact and U is open, $\Gamma \subset T^*\mathbb{R}^d$ a closed conic set. For all $f \in C^\infty(\Omega \times U, \mathbb{R})$ such that $\inf_{(x,a) \in \Omega \times A} |d_x f(\cdot, a)| > 0$ and $(\bigcup_{a \in A} \mathbf{Gr}d_x f(\cdot, a)) \cap \Gamma = \emptyset$:*

$$\exists \varphi \in \mathcal{D}(\Omega), \forall u \in \mathcal{D}'_\Gamma(\Omega), |\langle u, e^{i\tau f(\cdot, a)} \varphi \rangle| = O(\tau^{-\infty})$$

uniformly in $a \in A$.

Proof. — The idea of the proof comes from the following observation: given a function $f \in C^\infty(\Omega)$ such that $df|_\Omega \neq 0$, i.e. $f \in C^\infty(\Omega, \mathbb{R})$ is a submersion on Ω , then the concept of wave front set can be interpreted geometrically as the study of the regularity of the distribution $u \in \mathcal{D}'(\Omega)$ by **averaging on the level surfaces** of f . In mathematically precise terms, this will be given by the push-forward operation f_* . For all $\varphi \in \mathcal{D}(\Omega)$, the push-forward $f_*(u\varphi) \in \mathcal{D}'(\mathbb{R})$ is well defined and is the **Radon transform** of $u\varphi$ on the level surfaces of f (see [5] for the relationship of Radon transform with the wave front set). By definition of the Fourier transform

$$\widehat{f_*(u\varphi)}(\tau) = \int_{\mathbb{R}} dt (f_*(u\varphi))(t) e^{it\tau}$$

then

$$\widehat{f_*(u\varphi)}(\tau) = \langle u\varphi, f^*(e^{it\tau}) \rangle = \langle u\varphi, e^{i\tau f} \rangle$$

by definition of the pull-back as the adjoint of the push-forward.

If $WF(u)$ does not meet the graph of df in $T^*\Omega$ then by the push-forward theorem $WF(f_*(u\varphi)) \subset \{(f(x); \tau); (x; \tau df) \in WF(u) \cup \underline{Q}, \tau \neq 0\}$ but since $df \neq 0$ on Ω , we have the better estimate $WF(f_*(u\varphi)) \subset$

$\{(f(x); \tau) | (x; \tau df) \in WF(u) \cap (\dot{T}^*\Omega)\} = \emptyset$ since $WF(u) \cap \mathbf{Gr}df = \emptyset$. Therefore $WF(f_*(u\varphi))$ is empty and since $f_*(u\varphi)$ is compactly supported it is a smooth compactly supported function and its Fourier transform $\widehat{f_*(u\varphi)}(\tau)$ has fast decrease when $\tau \rightarrow +\infty$. To be completely faithful to the idea of Duistermaat, f should depend on some auxiliary parameter a which lives in a **compact space** $A \subset U$, i.e. $f \in C^\infty(\Omega \times U, \mathbb{R})$. Then our discussion is exactly the same except we should apply the parameter version of the push-forward theorem

$$f(\cdot, a)_*(u\varphi)(\tau) = \left\langle u\varphi, e^{i\tau f(\cdot, a)} \right\rangle,$$

if $\forall a \in A, \mathbf{Gr}d_x f(\cdot, a) \cap WF(u\varphi) = \emptyset$ then the parameter version of the push-forward theorem implies that the family $f(\cdot, a)_*(u\varphi)_{a \in A}$ is bounded in $\mathcal{D}'_\emptyset(\mathbb{R}) = C^\infty(\mathbb{R})$ uniformly in $a \in A$. \square

9. Acknowledgements

We are very grateful to Yoann Dabrowski for his generous help in the functional analysis parts of the paper. We thank Camille Laurent-Gengoux for discussions about the geometrical aspects of the pull-back.

10. Appendix: Technical results

This appendix gather different useful results. Several of them are folklore results for which we could find no proof in the literature.

10.1. Exhaustion of the complement of Γ

To prove that \mathcal{D}'_Γ is nuclear, we need the fact that the additional seminorms can be taken in a countable set. Therefore, we need to prove that the complement $\Gamma^c = \dot{T}^*M \setminus \Gamma$ of any closed cone $\Gamma \subset \dot{T}^*M$ can be exhausted by a countable set of products $U \times V$, where $U \subset M$ and V is a conic subset of \mathbb{R}^n .

First we introduce the sphere bundle (or unit cotangent bundle) $UT^*M = \{(x; k) \in T^*M; |k| = 1\}$, a vector bundle over M . We then define the set $U\Gamma_K = \{(x; k) \in \Gamma; x \in K, |k| = 1\} = UT^*M|_K \cap \Gamma$, for any compact subset $K \subset M$. Since M can be covered by a countable union of compact sets, we assume without loss of generality that K is covered

by a single chart (U, ψ) such that $\psi(K) \subset Q$, where $Q = [-1, 1]^n$ is an n -dimensional cube. We hence can assume w.l.g. that $M = \mathbb{R}^n$ and $UT^*M = \mathbb{R}^n \times S^{n-1}$. It will be convenient to use the norm d_∞ on \mathbb{R}^n , defined by: $\forall x, y \in \mathbb{R}^n$, $d_\infty(x, y) := \sup_{1 \leq i \leq n} |x_i - y_i|$. We denote by $\overline{B}_\infty(x, r) = \{y \in \mathbb{R}^n; d_\infty(x, y) \leq r\}$ the closed ball of radius r for this norm. We also denote the restriction of d_∞ to $S^{n-1} \times S^{n-1}$ by the same letter and, lastly, for $(x; \xi), (y; \eta) \in UT^*\mathbb{R}^n$ we set $d_\infty((x; \xi), (y; \eta)) = \sup(d_\infty(x, y), d_\infty(\xi, \eta))$.

We define cubes centered at rational points in Q : let $q_j = [-\frac{1}{2^j}, \frac{1}{2^j}]^n = \overline{B}_\infty(0, 2^{-j})$ and $q_{j,m} = \frac{m}{2^j} + q_j = \overline{B}_\infty(2^{-j}m, 2^{-j})$, where $m \in \mathbb{Z}^n \cap 2^j Q$. In other words, the center of $q_{j,m}$ runs over a hypercubic lattice with coordinates $(2^{-j}m_1, \dots, 2^{-j}m_n)$, where $-2^j \leq m_i \leq 2^j$. Note that, for each non-negative integer j , the hypercubes $q_{j,m}$ overlap and cover Q :

$$(10.1) \quad Q \subset \bigcup_{m \in \mathbb{Z}^n \cap 2^j Q} q_{j,m}.$$

Denote by $\underline{\pi} : UT^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\overline{\pi} : UT^*\mathbb{R}^n \longrightarrow S^{n-1}$ the projection maps defined by $\underline{\pi}(x; k) = x$ and $\overline{\pi}(x; k) = k$. We define $F_{j,m} = \underline{\pi}^{-1}(q_{j,m}) \simeq q_{j,m} \times S^{n-1}$ (see fig. 1). The set $\overline{\pi}(F_{j,m} \cap U\Gamma_K)$ is compact because the projection $\overline{\pi}$ is continuous and $F_{j,m} \cap U\Gamma_K$ is compact. For any positive integer ℓ , define the compact set

$$C_{j,m,\ell} = \{ \eta \in S^{n-1}; d_\infty(\overline{\pi}(F_{j,m} \cap U\Gamma_K), \eta) \geq 1/\ell \}.$$

This is the set of points of the sphere which are at least at a distance $1/\ell$ from the projection of the slice of UT inside $F_{j,m}$ (see fig. 1). We have

$$\bigcup_{\ell > 0} C_{j,m,\ell} = S^{n-1} \setminus \overline{\pi}(F_{j,m} \cap U\Gamma_K).$$

Indeed, by definition, any element of $C_{j,m,\ell}$ is in S^{n-1} and not in $\overline{\pi}(F_{j,m} \cap U\Gamma_K)$, conversely, the compactness of $\overline{\pi}(F_{j,m} \cap U\Gamma_K)$ implies that any point $(x; \xi)$ in $S^{n-1} \setminus \overline{\pi}(F_{j,m} \cap U\Gamma_K)$ is at a finite distance δ from $\overline{\pi}(F_{j,m} \cap U\Gamma_K)$. If we take $\ell > 1/\delta$, we have $(x; \xi) \in C_{j,m,\ell}$. Note that all $C_{j,m,\ell}$ are empty if $\overline{\pi}(F_{j,m} \cap U\Gamma_K) = S^{n-1}$.

With this notation we can now state

LEMMA 10.1. —

$$(10.2) \quad \bigcup_{j,m,\ell} q_{j,m} \times C_{j,m,\ell} = UT^*M|_K \setminus U\Gamma_K,$$

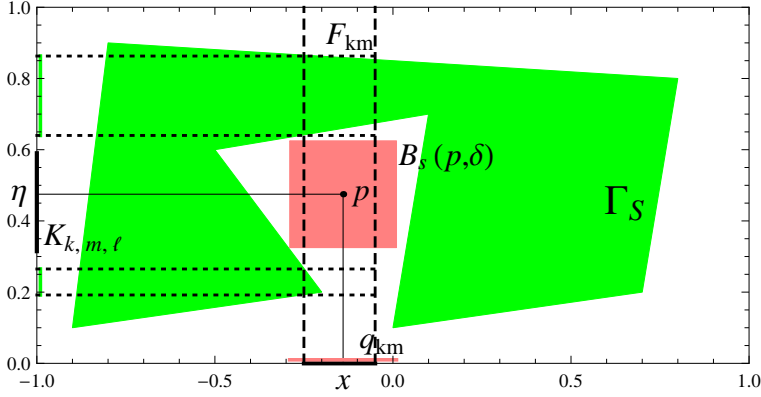


Figure 1. Picture of the case $n = 2$, where only one dimension of the cube $[-1, 1]^2$ is shown and the circle S^1 is represented by the vertical segment $[0, 1]$. The green surface is $U\Gamma_K$, the pink square is the ball $\overline{B}_\infty(p, \delta)$. We see that $q_{j,m}$ contains x and is contained in $\pi(\overline{B}_\infty(p, \delta))$.

and, by denoting $V_{j,m,\ell} = \{(x; k) \in \dot{T}^*M; (x; k/|k|) \in C_{j,m,\ell}\}$,

$$(10.3) \quad \bigcup_{j,m,\ell} q_{j,m} \times V_{j,m,\ell} = (T^*M \setminus \Gamma)|_K.$$

Proof. — We first prove the inclusion \subset in (10.2). Let $(x; k) \in \bigcup_{j,m,\ell} q_{j,m} \times C_{j,m,\ell}$, this means that there exist $j \in \mathbb{N}$, $m \in \mathbb{Z}^n \cap 2^j Q$ and $\ell \in \mathbb{N}^*$ such that $(x; k) \in q_{j,m} \times C_{j,m,\ell}$. Hence $(x; k) \in F_{j,m}$ and, by definition of $C_{j,m,\ell}$, $d_\infty(\pi(F_{j,m} \cap U\Gamma_K), k) \geq 1/\ell$, which implies that $(x; k) \notin U\Gamma_K$.

Let us prove the reverse inclusion \supset . Let $(x; k) \in UT^*M|_K \setminus U\Gamma_K$. Since this set is open, there exists some $\delta > 0$ such that $\overline{B}_\infty((x; k), \delta) \subset UT^*M|_K \setminus U\Gamma_K$. Let $j \in \mathbb{N}^*$ s.t. $2^{-j+1} < \delta$. Because the sets q_{jm} cover Q (see eq. (10.1)), there is an m such that $x \in q_{jm}$. Moreover, $\forall y \in q_{j,m}$, we have $d_\infty(x, y) \leq d_\infty(x, 2^{-j}m) + d_\infty(2^{-j}m, y) \leq 2^{-j} + 2^{-j} < \delta$, i.e. $y \in \overline{B}_\infty(x, \delta)$. Hence $q_{j,m} \subset \overline{B}_\infty(x, \delta)$. We deduce that

$$q_{j,m} \times \overline{B}_\infty(k, \delta) \subset \overline{B}_\infty(x, \delta) \times \overline{B}_\infty(k, \delta) = \overline{B}_\infty((x; k), \delta) \subset UT^*M|_K \setminus U\Gamma_K.$$

This means that $(q_{j,m} \times \overline{B}_\infty(k, \delta)) \cap U\Gamma_K = \emptyset$ or equivalently $F_{j,m} \cap \pi^{-1}(\overline{B}_\infty(k, \delta)) \cap U\Gamma_K = \emptyset$. The latter inclusion implies that $\overline{B}_\infty(k, \delta) \cap \pi(F_{j,m} \cap U\Gamma_K) = \emptyset$. In other words, $d_\infty(k, \pi(F_{j,m} \cap U\Gamma_K)) > \delta$. Hence by choosing $\ell \in \mathbb{N}^*$ s.t. $1/\ell \leq \delta$, we deduce that $d_\infty(k, \pi(F_{j,m} \cap U\Gamma_K)) > 1/\ell$,

i.e. that $k \in C_{j,m,\ell}$. Thus we conclude that $(x; k) \in q_{j,m} \times C_{j,m,\ell}$ and Eq. (10.2) is proved.

To prove (10.3), we notice that, because of the conic property of Γ , each $C_{j,m,\ell}$ corresponds to a unique $V_{j,m,\ell}$. \square

Any nonnegative smooth function ψ supported on $[-3/2, 3/2]^n$ and such that $\psi(x) = 1$ for $x \in [-1, 1]^n$ enables us to define scaled and shifted functions $\psi_{j-1,m}(x) = \psi(2^j(x - m))$ supported on $q_{j-1,m}$ and equal to 1 on $q_{j,2m}$.

If $C_{j,m,\ell}$ is not empty, we denote by $\alpha_{j,m,\ell} : S^{n-1} \rightarrow \mathbb{R}$ a smooth function supported on $C_{j,m,\ell}$ and equal to 1 on $C_{j,m,\ell+1}$. Note that, if $C_{j,m,\ell}$ is a proper subset of S^{n-1} , then it is strictly included in $C_{j,m,\ell+1}$.

10.2. Equivalence of topologies

Grigis and Sjöstrand stated [19, p. 80] that if we have a family χ_α of test functions and closed cones V_α such that $(\text{supp } \chi_\alpha \times V_\alpha) \cap \Gamma = \emptyset$ and $\cup_\alpha \{(x, k); \chi_\alpha(x) \neq 0 \text{ and } k \in \mathring{V}_\alpha\} = \Gamma^c$, then the topology of \mathcal{D}'_Γ is the topology given by the seminorms of the weak topology and the seminorms $\|\cdot\|_{N, V_\alpha, \chi_\alpha}$. By covering M with a countable family of compact sets K_i described in section 10.1, we see that Lemma 10.1 gives us a family of indices $\alpha = (i, j, \ell)$, functions $\chi_{j,m,\ell} = \psi_{j,m}$ and cones $V_{j,m,\ell}$ adapted to K_i such that the conditions of the Grigis-Sjöstrand lemma are satisfied. Therefore, the normal topology is described by the seminorms of the strong topology of $\mathcal{D}'(\Omega)$ and by the countable family (i, j, m, ℓ) of seminorms.

10.3. Topological equivalence $C^\infty(X)$ and \mathcal{D}'_\emptyset

As a first application of the previous lemma, we show

LEMMA 10.2. — *The spaces $C^\infty(X)$ and \mathcal{D}'_\emptyset are topologically isomorphic.*

Proof. — This property was also stated (without proof) by Alesker [1]. The two spaces are identical as vector spaces because a distribution u whose wave front set is empty is smooth everywhere, since its singular support $\text{sing supp}(u) = \pi(\text{WF}(u))$ [24, p. 254] is empty [24, p. 42], and a distribution is a smooth function if and only if its singular support is empty.

To prove the topological equivalence, we must show that the two inclusions $\mathcal{D}'_\emptyset \hookrightarrow C^\infty(X)$ and $C^\infty(X) \hookrightarrow \mathcal{D}'_\emptyset$ are continuous. Recall that a

system of semi-norms defining the topology of $C^\infty(X)$ is $\pi_{m,K}$, where m runs over the integers and K runs over the compact subsets of X [34, p. 88]. By a straightforward estimate, we obtain:

$$\begin{aligned} \pi_{m,K}(\varphi) &\leq C_n(2\pi)^{-n} \sum_{|\alpha| \leq m} \sup_{k \in \mathbb{R}^n} (1+|k|^2)^p |k^\alpha \widehat{\varphi \chi}(k)| \\ &\leq C_n(2\pi)^{-n} \binom{m+n}{n} \|\varphi\|_{m+2p, \mathbb{R}^n, \chi}, \end{aligned}$$

where $\chi \in \mathcal{D}(X)$ is equal to one on a compact set whose interior contains K where we used $(1+|k|^2) \leq (1+|k|)^2$, $|k^\alpha| \leq (1+|k|)^m$ and $\sum_{|\alpha| \leq m} = \binom{m+n}{n}$. Thus, every seminorm of $C^\infty(X)$ is bounded by a seminorm of \mathcal{D}'_\emptyset and the injection $\mathcal{D}'_\emptyset \hookrightarrow C^\infty(X)$ is continuous.

Conversely, for any closed conic set V and any $\chi \in \mathcal{D}(X)$ we have $\|\varphi\|_{N,V,\chi} \leq \|\varphi\|_{N,\mathbb{R}^n,\chi}$. Thus, it is enough to estimate $\|\varphi\|_{N,\mathbb{R}^n,\chi}$. We also find that, for any integer N and $\alpha = 0$,

$$\sup_{k \in \mathbb{R}^n} (1+k^2)^N |\widehat{\varphi \chi}(k)| \leq |K| 2^N \pi_{2N,K}(\varphi \chi),$$

where K is the support of φ . Then, the relation $1+|k| \leq 2(1+|k|^2)$ and application of the Leibniz rule give us $\|\varphi\|_{N,\mathbb{R}^n,\chi} \leq |K| 8^N \pi_{2N,K}(\chi) \pi_{2N,K}(\varphi)$ and the seminorms $\|\cdot\|_{N,V,\chi}$ are indeed controlled by the seminorms of $C^\infty(X)$.

It remains to show that this also holds for the seminorms of $\mathcal{D}'(X)$. It is well known that the inclusion $C^\infty(X) \hookrightarrow \mathcal{D}'(X)$ is continuous, but we can prove it for completeness. The weak topology of $\mathcal{D}'(X)$ is generated by the seminorms $p_f : u \mapsto |\langle u, f \rangle|$ where f runs over the test function space $\mathcal{D}(X)$. Let K be a compact neighborhood of $\text{supp } f$. We have, for every $\phi \in C^\infty(X)$,

$$p_f(\phi) = |\langle \phi, f \rangle| = \left| \int \phi(x) f(x) dx \right| \leq \pi_{0,K}(\phi) \pi_{0,K}(f) |K|,$$

where $|K|$ is the volume of K . Thus, the seminorms p_f are also controlled by those of $C^\infty(X)$. We can also consider the strong topology of $\mathcal{D}'(X)$, for which the seminorms are $p_B(\phi) = \sup_{f \in B} |\langle \phi, f \rangle|$, where B is a bounded set of $\mathcal{D}(X)$. Then, we have again $p_B(\phi) \leq \pi_{0,K}(\phi) M_0 |K|$, where K is a compact set containing the support of all functions of B and M_0 is the upper bound of $\pi_{0,K}(f)$ for all $f \in B$. Therefore, the inclusion $C^\infty(X) \hookrightarrow \mathcal{D}'_\emptyset$ is also continuous. We proved that $C^\infty(X)$ and \mathcal{D}'_\emptyset are topologically isomorphic, where \mathcal{D}'_\emptyset can be equipped with the Hörmander or the normal topology. \square

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